Appendix for "Offshoring in a Ricardian World"

This Appendix presents the proofs of Propositions 1 - 6 and the derivations of the results in Section IV.

Proof of Proposition 1

We want to show that

$$\left(\frac{T_m}{L_m}\right)^{\kappa}\left(T_m w_m^{-\theta}+\Phi_{-m}\right)^{1/\theta} \geq \left(\frac{T_1}{L_1}\right)^{\kappa}\left(T_1 w_1^{-\theta}+T_2 w_2^{-\theta}+\Phi_{-m}\right)^{1/\theta}$$

Since $\left(\frac{T_1}{L_1}\right)^{\kappa} > \left(\frac{T_m}{L_m}\right)^{\kappa}$, it is enough to prove the inequality for $\Phi_{-m}=0$. Thus, using $w_i=\delta\left(T_i/L_i\right)^{\kappa}$ for $i=1,2,m$ then we need to show that

$$\begin{aligned}
\left(\frac{T_m}{L_m}\right)^{\kappa}\left(T_m\delta^{-\theta}\left(\frac{L_m}{T_m}\right)^{\kappa\theta}\right)^{1/\theta} &= \frac{(T_m)^{1/\theta}}{\delta} \\
&\geq \left(\frac{T_1}{L_1}\right)^{\kappa}\left(T_1\delta^{-\theta}\left(\frac{L_1}{T_1}\right)^{\kappa\theta}+T_2\delta^{-\theta}\left(\frac{L_2}{T_2}\right)^{\kappa\theta}\right)^{1/\theta} \\
&= \left(\frac{T_1}{L_1}\right)^{\kappa}\frac{\left(T_1\left(\frac{L_1}{T_1}\right)^{\kappa\theta}+T_2\left(\frac{L_2}{T_2}\right)^{\kappa\theta}\right)^{1/\theta}}{\delta}
\end{aligned}$$

We have

$$\left(\frac{T_1}{L_1}\right)^{\kappa}\frac{\left(T_1\left(\frac{L_1}{T_1}\right)^{\kappa\theta}+T_2\left(\frac{L_2}{T_2}\right)^{\kappa\theta}\right)^{1/\theta}}{\delta} \geq \left(\frac{T_1}{L_1}\right)^{\kappa}\frac{\left(T_1\left(\frac{L_1}{T_1}\right)^{\kappa\theta}+T_2\left(\frac{L_1}{T_1}\right)^{\kappa\theta}\right)^{1/\theta}}{\delta} = \frac{(T_m)^{1/\theta}}{\delta}.$$

Q.E.D.

Proof of Proposition 2

We have

$$\begin{aligned}
w_1(\alpha) &= (1+\alpha)\tilde{w}_1(\alpha)-\alpha w_2(\alpha) \\
&= \delta\left(\frac{T_1}{L_1}\right)^{\kappa}(1+\alpha)^{1-\kappa}-\delta\left(T_2\right)^{\kappa}\frac{\alpha}{\left(L_2-\alpha L_1\right)^{\kappa}}
\end{aligned}$$

This implies that

$$w_1'(\alpha)=\delta\left(\frac{T_1}{L_1}\right)^{\kappa}\frac{(1-\kappa)}{(1+\alpha)^{\kappa}}-\frac{\delta\left(T_2\right)^{\kappa}}{\left(L_2-\alpha L_1\right)^{\kappa}}\left(1+\frac{\alpha\kappa L_1}{L_2-\alpha L_1}\right)$$

Obviously $\delta\left(\frac{T_1}{L_1}\right)^{\kappa}\frac{(1-\kappa)}{(1+\alpha)^{\kappa}}$ is decreasing, while $\frac{\delta(T_2)^{\kappa}}{(L_2-\alpha L_1)^{\kappa}}\left(1+\frac{\alpha\kappa L_1}{L_2-\alpha L_1}\right)$ is increasing in α. To determine the sign of $w_1'(\alpha)$ on $[0,\bar{\alpha}]$ we should then compare

$w_1'(0)$ and $w_1'(\bar{\alpha})$ with zero. Focusing first on $w_1'(\bar{\alpha})$, using the definition of $\bar{\alpha}$, we get

$$w_1'(\bar{\alpha}) = \frac{\kappa T_2^{\kappa} \delta \left(1 + \eta L_1/L_2\right)^{\kappa}}{\left(L_2 + L_1\right)^{\kappa}} \left(-1 - \frac{\left(\eta - 1\right) L_1}{L_2 + L_1}\right) < 0$$

Turning to $w_1'(0)$, note that

$$\begin{aligned} w_1'(0) &= \delta \left(\frac{T_1}{L_1}\right)^{\kappa} (1 - \kappa) - \frac{\delta T_2^{\kappa}}{L_2^{\kappa}} \\ &= \delta \left(\frac{T_2}{L_2}\right)^{\kappa} \left(\eta^{\kappa} \left(1 - \kappa\right) - 1\right) > 0 \iff \eta > \left(1 - \kappa\right)^{-1/\kappa} \end{aligned}$$

Thus, if $\eta \leq (1-\kappa)^{-1/\kappa}$, then $w_1(\alpha)$ is always decreasing on $[0, \bar{\alpha}]$. If $\eta > (1-\kappa)^{-1/\kappa}$, then $w_1(\alpha)$ is shaped like an inverted U on $[0, \bar{\alpha}]$. **Q.E.D.**

Proof of Proposition 3

Recall that $\Phi \equiv \sum_k T_k c_k^{-\theta}$. Thus, it is useful to use

$$\Phi = T_1 c_1^{-\theta} + T_2 w_2^{-\theta} + \Phi_{-m}$$

where Φ_{-m} is not affected by α. We know that $c_1 = \delta \left(T_1/\tilde{L}_1\right)^{\kappa}$ and $w_2 = \delta \left(T_2/\tilde{L}_2\right)^{\kappa}$, so

$$T_1 c_1^{-\theta} + T_2 w_2^{-\theta} = \delta^{-\theta} \left(T_1^{\kappa} L_1^{\theta\kappa} (1+\alpha)^{\theta\kappa} + T_2^{\kappa} (L_2 - \alpha L_1)^{\theta\kappa}\right)$$

This implies that

$$\left(T_1 c_1^{-\theta} + T_2 w_2^{-\theta}\right)_{\alpha}' = \delta^{-\theta} \theta \kappa L_1 \left[\left(T_1/L_1\right)^{\kappa} \left(1+\alpha\right)^{-\kappa} - T_2^{\kappa} (L_2 - \alpha L_1)^{-\kappa}\right]$$

We need to compare $f(\alpha) \equiv \left(\frac{T_1}{L_1}\right)^{\kappa} (1+\alpha)^{-\kappa} - T_2^{\kappa} (L_2 - \alpha L_1)^{-\kappa}$ with zero on $[0, \bar{\alpha})$. Obviously, $f(0) = \left(\frac{T_1}{L_1}\right)^{\kappa} - \left(\frac{T_2}{L_2}\right)^{\kappa} > 0$, while simple algebra reveals that $f(\bar{\alpha}) = 0$. Since $f'(\alpha) < 0$, then $f(\bar{\alpha}) = 0$ implies that $f(\alpha) > 0$ for any $\alpha \in [0, \bar{\alpha}[$. This means that $\left(T_1 c_1^{-\theta} + T_2 w_2^{-\theta}\right)_{\alpha}' > 0$, or $\Phi_{\alpha}' > 0$. But given $P = \gamma \Phi^{-1/\theta}$ then this implies that $P_{\alpha}' < 0$. **Q.E.D.**

Proof of Proposition 4

We know that the sign of $\left(\frac{w_1}{P}\right)_{\alpha}'$ is the same as the sign of $\frac{w_1'}{w_1} - \frac{P'}{P}$. But simple differentiation and simplification reveals that

$$\begin{aligned} \frac{w_1'}{w_1} &= G(x, \alpha) \equiv \frac{x(1-\kappa)\left(\frac{f(\alpha)}{1+\alpha}\right)^{\kappa} - \left(1 + \frac{\alpha \kappa L_1/L_2}{f(\alpha)}\right)}{x(1+\alpha)\left(\frac{f(\alpha)}{1+\alpha}\right)^{\kappa} - \alpha} \\ \frac{P'}{P} &= F(x, \alpha) \equiv -\kappa \frac{x \frac{1}{(1+\alpha)^{\kappa}} - \frac{1}{(f(\alpha))^{\kappa}}}{x(1+\alpha)^{\theta\kappa} + \frac{L_2}{L_1}(f(\alpha))^{\theta\kappa} + \frac{\delta^{\theta}\Phi^*}{L_1}} \end{aligned}$$

where $x \equiv \eta^\kappa$, $f(\alpha) = 1 - \alpha L_1/L_2$. Let $x_F(\alpha)$ and $x_G(\alpha)$ be defined implicitly by $F(x,\alpha) = 0$ and $G(x,\alpha) = 0$, respectively. The following lemma, whose proof is simple and therefore omitted, summarizes a number of properties of these functions:

Lemma 1 *$F(x,\alpha)$ is decreasing in x, $G(x,\alpha)$ is increasing in x,*

$$x_F(\alpha) = \left(\frac{1+\alpha}{f(\alpha)}\right)^\kappa > 1, \quad \text{and } x_G(\alpha) = \frac{\left(1 + \frac{\alpha\kappa L_1/L_2}{f(\alpha)}\right)}{(1-\kappa)\left(\frac{f(\alpha)}{1+\alpha}\right)^\kappa} > 1.$$

Also, $x_F(\alpha) < x_G(\alpha)$, $x'_F(\alpha) > 0$, $x'_G(\alpha) > 0$.

Let $x_M(\alpha)$ be defined implicitly by $G(x,\alpha) = F(x,\alpha)$. Such a solution necessarily exists since $x_F(\alpha) < x_G(\alpha)$ and $F(x,\alpha)$ is decreasing in x and $G(x,\alpha)$ is increasing in x. Also, it is clear that $1 < x_F(\alpha) < x_M(\alpha) < x_G(\alpha)$. Since $x > x_M(\alpha)$ implies $G > F$ then it also implies that w_1/P is increasing. Similarly, $x < x_M(\alpha)$ implies that w_1/P is decreasing. The following lemma (whose is proof is long and tedious and therefore provided at the end of this Appendix) is critical:

Lemma 2 *$x_M(\alpha)$ is increasing*

Let $\hat{\eta}$ be equal to $x_M(0)^{1/\kappa}$. If $\eta \leq \hat{\eta}$, then $x = \eta^\kappa \leq x_M(0) \leq x_M(\alpha)$ for any α. This implies that $F(x) > G(x)$ (except the case when $x = \hat{\eta}^\kappa$ and $\alpha = 0$), so w_1/P is decreasing. This establishes the first part of the proposition. To establish the second part, we need the following lemma:

Lemma 3 *For any $\eta > \hat{\eta} = (x_M(0))^{1/\kappa}$ we have $x = \eta^\kappa < x_M(\bar{\alpha}(\eta))$.*

Proof. The proof relies on showing that $F(\eta^\kappa, \bar{\alpha}(\eta)) = 0$, which implies that $\eta^\kappa = x_F(\bar{\alpha}(\eta))$. If this is true then $x_M(\bar{\alpha}(\eta)) > \eta^\kappa$, because since $x_F(\alpha) < x_M(\alpha)$ for all α then $\eta^\kappa = x_F(\bar{\alpha}(\eta)) < x_M(\bar{\alpha}(\eta))$, which establishes the result. But from the definition of $\bar{\alpha}$ we see that

$$\eta = \frac{1+\bar{\alpha}}{f(\bar{\alpha})}$$

and plugging this into $F(\eta^\kappa, \bar{\alpha})$ shows that $F(\eta^\kappa, \bar{\alpha}(\eta)) = 0$. ■

This lemma implies that if $\eta > \hat{\eta}$ then w_1/P is increasing for $\alpha = 0$ and decreasing just before $\alpha = \bar{\alpha}(\eta)$, with a unique point α for which $x_M(\alpha) = x$ at which $G = F$ and hence $(w_1/P)'_\alpha = 0$. This implies that the curve w_1/P as a function of α in the interval $\alpha \in [0, \bar{\alpha}]$ is shaped like an inverted U.

Proof of Proposition 5

The only thing left to show is that steady state P is decreasing in α. It is sufficient to show that $\Phi_{mt} = T_{1t}c_{1t}^{-\theta} + T_{2t}w_{2t}^{-\theta}$ is decreasing in α. But

$$\Phi_{mt} = (1/L_{2t}^F g_L)\left(\phi_1 r_1 \varphi c_{1t}^{-\theta} + \phi_2 r_2 w_2^{-\theta}\right)$$

Using $c_1^{1-\kappa} = \left(\frac{\phi_1/\phi_N}{w_1}\right)^{\kappa}$ and $w_2 = (\phi_2/\phi_N)^{\kappa}$ then

$$\begin{aligned}\Phi_{mt} &= (1/L_{2t}^F g_L)\left(\phi_1 r_1 \varphi \left(\frac{\phi_1/\phi_N}{w_{1t}}\right)^{-\kappa\theta/(1-\kappa)} + \phi_2 r_2 \left(\phi_2/\phi_N\right)^{-\kappa\theta}\right) \\ &= (\phi_N/L_{2t}^F g_L)\left(\varphi r_1 w_1 + w_2 r_2\right)\end{aligned}$$

But plugging in from the equations (24) and (25) we get that

$$\varphi r_1 w_1 + w_2 r_2 = \varphi w_1 + w_2$$

which is increasing in α. **Q.E.D.**

Proof of Proposition 6

I first show that $x = \alpha w_2/w_1$ is increasing in α. From (29) we get $(\phi_1/\phi_N)^{\kappa} = z\,(1+x)^{1-\kappa} = z^{\kappa}(z + w_2\beta(1-\beta)^{-\kappa})^{1-\kappa}$, where $z \equiv (1-\beta)^{1-\kappa}w_1$. Since $\beta(1-\beta)^{-\kappa}$ is increasing in α then z must be decreasing in α. In turn, this implies that x must be increasing in α.

Now, recall that r_1 is determined as the solution of $r_1 = r\left(1 + \alpha\left(1 - r_1\right) w_2/w_1\right)$. Both the LHS and the RHS are linear functions in r_1, with the LHS increasing and the RHS decreasing. An increase in α moves the RHS schedule upward because $\alpha w_2/w_1$ increases with α, while the LHS schedule remains the same. This implies that r_1 increases.

In the text, before stating proposition **6** I also stated that $\alpha(1-r_1)w_2/w_1$ is increasing in α. To see this, note that since r_1 is increasing in alpha, then the RHS of $r_1 = r\left(1 + \alpha\left(1 - r_1\right) w_2/w_1\right)$ must be increasing in α, so $\alpha\left(1 - r_1\right) w_2/w_1$ is increasing in α.

Finally, to prove that r_2 is decreasing in α, from (25) I need to show that $\alpha(1-r_1)$ is increasing in α. But we know that $\alpha(1-r_1)w_2/w_1$ is increasing in α while w_2 is constant and w_1 is increasing. This implies that $\alpha(1-r_1)$ must be increasing in α. **Q.E.D.**

Equilibrium with offshoring costs: 2 countries

Let

$$C(w_1, w_2, \lambda) \equiv w_2 F(1,\lambda) + \int_{w_2}^{w_1} x dF(x, \lambda/w_2)dx + w_1(1 - F(w_1, \lambda/w_2))$$

Integration and simplification yields

$$C(w_1, w_2, \lambda) = w_2 - (w_2/\lambda)\exp(-\lambda w_1/w_2) + (w_2/\lambda)\exp(-\lambda)$$

Letting $s(w_1, w_2, \lambda) \equiv F(w_1/w_2, \lambda)$ and $\sigma(w_1, w_2, \lambda) = \Gamma(w_1, w_2, \lambda)s(w_1, w_2, \lambda)$, or

$$\sigma(w_1, w_2, \lambda) \equiv 1 - (1/\lambda)\exp(-\lambda w_1/w_2) - w_1/w_2 \exp(-\lambda w_1/w_2) + (1/\lambda)\exp(-\lambda)$$

then an equilibrium for $\lambda < \lambda_m$ is determined by the following equations:

$$C(w_1, w_2, \lambda) = \delta \left(\frac{T_1}{L_1/(1 - s(w_1, w_2, \lambda))} \right)^{\kappa} \tag{1}$$

and

$$w_2 = \delta \left(\frac{T_2}{L_2 - \sigma(w_1, w_2, \lambda) L_1/(1 - s(w_1, w_2, \lambda))} \right)^{\kappa} \tag{2}$$

Totally differentiating above yields equation (1) yields

$$\frac{dC}{d\lambda} = \frac{\partial C}{\partial w_1}\frac{dw_1}{d\lambda} + \frac{\partial C}{\partial w_2}\frac{dw_2}{d\lambda} + \frac{\partial C}{\partial \lambda} = -\delta (T_1/L_1)^{\kappa} \kappa (1-s)^{\kappa - 1} \frac{ds}{d\lambda}$$

As $\lambda \to 0$ then $s \to 0$, hence $\partial C / \partial w_2 \to 0$ and

$$\frac{\partial C}{\partial w_1}\frac{dw_1}{d\lambda} + \frac{\partial C}{\partial \lambda} = -\delta (T_1/L_1)^{\kappa} \kappa \frac{ds}{d\lambda}$$

Now,

$$\begin{aligned} \frac{ds}{d\lambda} &= \frac{\partial s}{\partial w_1}\frac{dw_1}{d\lambda} + \frac{\partial s}{\partial w_2}\frac{dw_2}{d\lambda} + \frac{\partial s}{\partial \lambda} \\ &= (\lambda / w_2) \exp(-\lambda w_1 / w_2) - \left(\lambda w_1 / w_2^2 \right) \exp(-\lambda w_1 / w_2) + (w_1/w_2) \exp(-\lambda w_1 / w_2) \end{aligned}$$

When $\lambda \to 0$ then $ds/d\lambda \to w_1/w_2$. Noting that $w_1 \to \delta (T_1/L_1)^{\kappa}$ as $\lambda \to 0$, we see that

$$\frac{\partial C}{\partial w_1}\frac{dw_1}{d\lambda} = -\kappa \frac{w_1^2}{w_2} - \frac{\partial C}{\partial \lambda}$$

Simple differentiation reveals that

$$\frac{\partial C}{\partial \lambda} = \frac{w_2 \exp(-\lambda w_1/w_2) - w_2 \exp(-\lambda) + w_1 \lambda \exp(-\lambda w_1 / w_2) - w_2 \lambda \exp(-\lambda)}{\lambda^2}$$

Since both the numerator and denominators converge to 0 as $\lambda \to 0$ then we can use L'Hopital's Theorem to find that

$$\begin{aligned} \lim_{\lambda \to 0} \frac{\partial C}{\partial \lambda} &= \lim_{\lambda \to 0} \frac{w_1(-\lambda)\,(w_1/w_2) \exp(-\lambda w_1/w_2) + w_2 \lambda \exp(-\lambda)}{2\lambda} \\ &= \frac{w_2 - w_1^2 / w_2}{2} \end{aligned}$$

Thus,

$$\begin{aligned} \lim \left(-\kappa \frac{w_1^2}{w_2} - \frac{\partial C}{\partial \lambda} \right) &= -\kappa w_2 \left(\frac{w_1}{w_2} \right)^2 - (w_2/2) \left(1 - \left(\frac{w_1}{w_2} \right)^2 \right) \\ &= (w_2/2) \left[(w_1/w_2)^2 (1 - 2\kappa) - 1 \right] \end{aligned}$$

This implies that $\lim_{\lambda \to 0} dw_1/d\lambda > 0$ if and only if $(w_1/w_2)^2 > 1/(1 - 2\kappa)$.

To show that $\partial C/\partial\lambda < 0$, note that this is equivalent to

$$(1 + w_1\lambda/w_2)\exp(-\lambda w_1/w_2) < (1 + \lambda)\exp(-\lambda)$$

This inequality holds since the function $f(x) \equiv (1 + x)\exp(-x)$ is clearly decreasing.

Equilibrium with offshoring costs: 3 countries

I first characterize an equilibrium in which $w_1 > w_3 > w_2$. In this case the distribution of c_1 is

$$\Pr(C_1 \leq c_1) = \begin{cases} 0 & if \quad c_1 < w_2 \\ F(c_1, \lambda/w_2) & if \quad c_1 \in [w_2, w_3[\\ F(c_1, \varphi) & if \quad c_1 \in [w_3, w_1[\\ 1 & if \quad c_1 \geq w_1 \end{cases}$$

where $\varphi \equiv \lambda/w_2 + \lambda/w_3$, and unit cost of the common input in country 1 is

$$\begin{aligned} c_1(w_1, w_2, w_3) \quad \equiv \quad & w_2 F(1, \lambda) + \int_{w_2}^{w_3} dF(x, \lambda/w_2) \\ & + w_3\left(F(w_3, \varphi) - F(w_3, \lambda/w_2)\right) + \int_{w_3}^{w_1} dF(x, \varphi) + w_1(1 - F(w_1, \varphi)) \end{aligned}$$

The equilibrium condition for country 1 is

$$c_1(w_1, w_2, w_3) = \delta\left(\frac{T_1(1 - s}{L_1}\right)^{\kappa}$$

where $s = 1 - \exp(-\varphi w_1)$ is the total share of services offshored. This share is distributed between countries 2 and 3 as follows:

$$s_{12} = F(w_3, \lambda/w_2) + [F(w_1, \varphi) - F(w_3, \varphi)]\left(\frac{\lambda/w_2}{\varphi}\right)$$

and

$$s_{13} = F(w_3, \lambda/w_2 + \lambda/w_3) - F(w_3, \lambda/w_2) + [F(w_1, \varphi) - F(w_3, \varphi)]\left(\frac{\lambda/w_3}{\varphi}\right)$$

Note that $s = s_{12} + s_{13}$.

On the other hand, we have that

$$\Pr(C_3 \leq c_3) = \begin{cases} 0 & if \quad c_3 < w_2 \\ F(c_3, \lambda/w_2) & if \quad c_3 \in [w_2, w_3[\\ 1 & if \quad c_3 \geq w_3 \end{cases}$$

and

$$c_3(w_1, w_2, w_3) \equiv w_2 F(1, \lambda) + \int_{w_2}^{w_3} x dF(x, \lambda_2/w_2) + w_3(1 - F(w_3, \lambda_2/w_2)$$

and

$$s_{32}(w_2, w_3) = F(w_3, \lambda_2/w_2)$$

The equilibrium conditions for countries 2 and 3 are $c_i(w_1, w_2, w_3) = \delta\left(T_i/\widetilde{L}_i\right)^{\kappa}$ for $i = 2, 3$, with $c_2(w_1, w_2, w_3) = w_2$. To derive $\widetilde{L}_3$, note that country 1 uses $L_1/(1-s)$ of every service, and if the offshoring cost of a service in country 3 is ζ_3 then it takes $\zeta_3 L_1/(1-s)$ units of labor to produce $L_1/(1-s)$ units of a service delivered in country 1. The expectation of ζ_3 for services offshored by country 1 to country 3 is

$$\sigma_{13}(w_1, w_2, w_3) \equiv F(w_3, \varphi) - F(w_3, \lambda/w_2) + (1/w_3)\left(\frac{\lambda/w_3}{\varphi}\right)\int_{w_3}^{w_1} x dF(x, \varphi)$$

This implies that $L_3 - \left(\frac{\sigma_{13}}{1-s}\right)L_1$ units of labor are left in country 3 for final good production. But this country will offshore s_{32} services to country 2, so it will use $\left(L_3 - \left(\frac{\sigma_{13}}{1-s}\right)L_1\right)\left(\frac{1}{1-s_{32}}\right)$ of each service for domestic production. This implies that

$$\widetilde{L}_3 = \left(L_3 - \left(\frac{\sigma_{13}}{1-s}\right)L_1\right)\left(\frac{1}{1-s_{32}}\right)$$

On the other hand, country 2 export services to countries 1 and 3. The labor it takes to export services to country 1 is $\left(\frac{\sigma_{12}}{1-s}\right)L_1$, where σ_{12} is the expectation of ζ_2 for services offshored by country 1 to country 2, and is given by

$$\begin{aligned}\sigma_{12}(w_1, w_2, w_3) \quad \equiv \quad & (1 - \exp(-\lambda)) + (1/w_2)\int_{w_2}^{w_3} x dF(x, \lambda/w_2) \\ & + (1/w_2)\left(\frac{\lambda/w_2}{\varphi}\right)\int_{w_3}^{w_1} x dF(x, \varphi)\end{aligned}$$

Country 2 also exports services to country 3, and by the reasoning above, we know that it takes $\left(L_3 - \left(\frac{\sigma_{13}}{1-s}\right)L_1\right)\left(\frac{\sigma_{32}}{1-s_{32}}\right)$ units of labor to do so, with σ_{32} representing the expectation of ζ_2 for services offshored by country 3 to country 2, given by

$$\sigma_{32}(w_1, w_2, w_3) \equiv (1 - \exp(-\lambda)) + (1/w_2)\int_{w_2}^{w_3} x dF(x, \lambda/w_2)$$

Thus, we find that

$$\widetilde{L}_2 = L_2 - \left(\frac{\sigma_{12}}{1-s}\right)L_1 - \left(L_3 - \left(\frac{\sigma_{13}}{1-s}\right)L_1\right)\left(\frac{\sigma_{32}}{1-s_{32}}\right)$$

If the previous system yields wages that do not respect $w_1 > w_3 > w_2$ then it is not an equilibrium. Other possible equilibrium configurations have

$w_1 > w_2 = w_3$, $w_1 = w_3 > w_2$, and $w_1 = w_2 = w_3$. The last case entails full offshoring. Such an equilibrium satisfies

$$\frac{T_m}{L_m} = \frac{T_1(1-s)}{L_1} = \frac{T_2}{L_2 - \left(\frac{s_{12}}{1-s}\right) L_1 - \left(L_3 - \left(\frac{s_{13}}{1-s}\right) L_1\right)\left(\frac{s_{32}}{1-s_{32}}\right)} = \frac{T_3(1-s_{32})}{L_3 - \left(\frac{s_{13}}{1-s}\right) L_1}$$

with $s = s_{12} + s_{13}$. For every s_{32} these equations determine s_{12} and s_{13}, so these variables are not uniquely pinned down in equilibrium: there is no uniqueness because of the absence of offshoring costs for the services that are traded, but all equilibria entail the same wages.[1] If one can find a solution with $s < F(1, 2\lambda)$ and $s_{12}, s_{13}, s_{32} < F(1, \lambda)$, then this solution corresponds to an equilibrium with full offshoring. Since $F(1, 2\lambda)$ and $F(1, \lambda)$ both converge to 1 as $\lambda \to \infty$ then necessarily there is some critical value of λ such that for higher values of λ the equilibrium entails full offshoring.

I now establish the equilibrium conditions when wages entail $w_1 > w_2 = w_3$ and $w_1 = w_3 > w_2$. If $w_1 > w_2 = w_3$ then countries 2 and 3 are integrated and their wage should be the same as the one we would would get in a two country system, with

$$w_2 = w_3 = w_{23} = \delta \left(\frac{T_2 + T_3}{L_2 + L_3 - \left(\frac{\sigma}{1-s}\right) L_1} \right)^{\kappa}$$

but with countries 2 and 3 drawing their offshoring cost from $F(\zeta, 2\lambda)$, and with $s = F(w_1/w_{23}, 2\lambda)$ and

$$\sigma = F(1, 2\lambda) + \int_1^{w_1/w_{23}} x dF(x, 2\lambda)$$

This equilibrium has σ_{12} and σ_{13} and s_{32} such that

$$w_{23} = \delta \left(\frac{T_2}{L_2 - \left(\frac{\sigma_{12}}{1-s}\right) L_1 - \left(L_3 - \left(\frac{\sigma_{13}}{1-s}\right) L_1\right)\left(\frac{s_{32}}{1-s_{32}}\right)} \right)^{\kappa}$$

and

$$w_{23} = \delta \left(\frac{T_3}{L_3 - \left(\frac{\sigma_{13}}{1-s}\right) L_1 \left(\frac{1}{1-s_{32}}\right)} \right)^{\kappa}$$

with the restriction that $\sigma_{32} = s_{32} \leq F(1, \lambda)$ and

$$\sigma_{12}, \sigma_{13} \leq F(1, \lambda) + (1/2) \int_1^{w_1/w_{23}} x dF(x, 2\lambda)$$

[1] Although there are 3 equations for 3 unknowns (s_{12}, s_{13} and s_{32}), these equations are linearly dependent, so they determine only two unknowns.

The first term on the RHS is the measure of services for which ζ_2 or ζ_3 are equal to 1, while the second term is the offshoring cost for services with $\zeta_i \in]1, w_1/w_3]$.

Now consider the case with $w_1 = w_3 > w_2$. This entails

$$w_1 = w_3 = w_{13} = \delta \left(\frac{(T_1 + T_3)(1 - s)}{L_1 + L_3} \right)^{\kappa}$$

and

$$w_2 = \delta \left(\frac{T_2}{L_2 - \left(\frac{\sigma}{1-s} \right)(L_1 + L_3)} \right)^{\kappa}$$

where $s = F(w_{13}/w_2, \lambda)$ and

$$\sigma = F(1, \lambda) + \int_1^{w_{13}/w_2} x dF(x, \lambda)$$

For this to be an equilibrium, we need that $\sigma_{13} = s_{13} \leq F(1, \lambda)$, have $s_{32} = s_{12} = F(w_{13}/w_2, \lambda)$ and

$$\sigma_{32} = \sigma_{12} = F(1, \lambda) + \int_1^{w_{13}/w_2} x dF(x, \lambda)$$

and the equations of the full system.

For the long run equilibrium, we have $c_i(w_1, w_2, w_3)^{1-\kappa} = \left(\frac{\phi_i/\phi_N}{w_i} \right)^{\kappa}$ for $i = 1, 2, 3$, with $c_2(w_1, w_2, w_3) = w_2$. This equation for $i = 2$ can be solved directly to yield $w_2 = (\phi_2/\phi_N)^{\kappa}$, as in (28). Plugging this into the equation for $i = 3$ yields an equation that can be solved for the equilibrium wage in country 3, w_3. We can easily check that w_3 is increasing in λ. Finally, plugging the solution $w_3(\lambda)$ into the equation for $i = 1$ yields the equilibrium $w_1(\lambda)$. I have not been able to show that $w_1(\lambda)$ is increasing.

Proof of Lemma 2

Lemma 2 above establishes that $x_M(\alpha)$ is increasing. To prove this lemma, I first introduce some notation. Let

$$\begin{aligned} H(x, \alpha) &= x^2 + Bx/A - C/A^2 \\ J(x, \alpha) &= \left[\left(1 + \frac{\alpha b L_1/L_2}{f(\alpha)} \right) - Ax(1 - b) \right] \frac{const}{(1 + \alpha)A^2} \end{aligned}$$

where

$$\begin{aligned} A &= \left(\frac{f(\alpha)}{1 + \alpha} \right)^b, \; const = \delta^{\theta} \left(f(\alpha) \right)^b \Phi_{-m}/L_1 \\ B &= (1 - b)C - \left(1 + \frac{\alpha b L_1/L_2}{f(\alpha)} \right) - b - \frac{\alpha}{(1 + \alpha)} b \\ C &= (L_2/L_1) \frac{f(\alpha)}{(1 + \alpha)} \end{aligned}$$

Simple algebra shows that $G(x,\alpha) = F(x,\alpha) \Leftrightarrow H(x,\alpha) = J(x,\alpha)$, so $x_M(\alpha)$ solves

$$H(x,\alpha) = J(x,\alpha)$$

The proof that $x_M(\alpha)$ is increasing includes three steps:

1) First, I prove that the solution $x_M^0(\alpha)$ of $H(x,\alpha) = 0$ is increasing in alpha. Since $J(x,\alpha)$ is flat in x if $\Phi_{-m} = 0$ (since $const = 0$) then this implies that if $\Phi_{-m} = 0$ then $x_M(\alpha) = x_M^0(\alpha)$ is increasing in α. The rest of the proof extends this to $\Phi^* > 0$.

2) Next, I prove that if $\alpha_2 > \alpha_1$ then $H(x,\alpha_2) < H(x,\alpha_1)$ for any $x \geq x_M^0(\alpha_1)$.

3) Finally, I prove that the solution of $J(x,\alpha_2) = J(x,\alpha_1)$, where α_2 is greater and close to α_1, is less than $x_M^0(\alpha_1)$.

Thus, given that the slope of J w.r.t. x increases (declines in absolute value) as α increases, then the three steps above are sufficient to prove that $x_M(\alpha)$ is increasing in alpha, since the shift of $J(x,\alpha)$ with an increase in alpha amplifies the effect of increasing α on $x_M^0(\alpha)$.

First step: We want to prove that $x_M^0(\alpha)$ is increasing in alpha. This is done by solving explicitly for the highest solution to $H(x,\alpha) = 0$ and then differentiating w.r.t. α and showing that the result is positive. Given the expression for $H(x,\alpha) = 0$ then $x_M^0(\alpha)$ is determined by the positive solution of

$$A^2x^2 + ABx - C = 0,$$

or

$$x_M(\alpha) = \frac{-B + \sqrt{B^2 + 4C}}{2A}$$

Differentiation yields:

$$\frac{dx_M(\alpha)}{d\alpha} = \frac{A\left(\frac{2BB' + 4C'}{2\sqrt{B^2+4C}} - B'\right) - A'\left(\sqrt{B^2 + 4C} - B\right)}{2A^2}.$$

It is easy to show that this is positive if and only if

$$(A'B - AB')\left(\sqrt{B^2 + 4C} - B\right) > A'4C - 2C'A$$

Differentiating to get A' and C' and then plugging in and simplifying reveals that

$$A'4C - 2C'A = 2A\frac{1 + L_2/L_1}{(1+\alpha)^2}(1 - 2b).$$

Hence, we want to show that

$$\left(\frac{A'}{A}B - B'\right)\left(\sqrt{B^2 + 4C} - B\right) > 2\frac{1 + L_2/L_1}{(1+\alpha)^2}(1 - 2b)$$

Now,

$$\frac{A'}{A}=\frac{-b\left(\frac{f(\alpha)}{(1+\alpha)}\right)^{b-1}\frac{1+L_1/L_2}{(1+\alpha)^2}}{\left(\frac{f(\alpha)}{(1+\alpha)}\right)^{b}}=-b\frac{1+L_1/L_2}{f(\alpha)\,(1+\alpha)}$$

and

$$-B'=(1-b)(L_2/L_1)\frac{1+L_1/L_2}{(1+\alpha)^2}+bL_1/L_2\frac{1}{(f(\alpha))^2}+\frac{b}{(1+\alpha)^2}.$$

Consider $\sqrt{B^2+4C}-B$ as a function of $b\in(0,1/2)$. We have

$$\begin{aligned}\left(\sqrt{B^2+4C}-B\right)'_b &= \frac{2BB'}{2\sqrt{B^2+4C}}-B'\\ &= B'\left(\frac{B-\sqrt{B^2+4C}}{\sqrt{B^2+4C}}\right)>0,\end{aligned}$$

as $B-\sqrt{B^2+4C}<0$ and $B'<0$. Thus, it is sufficient to show that

$$\left(\frac{A'}{A}B-B'\right)\left(\sqrt{B^2+4C}-B\right)_{b=0}>2\frac{1+L_2/L_1}{(1+\alpha)^2}(1-2b),$$

But

$$\begin{aligned}\left(\sqrt{B^2+4C}-B\right)_{b=0} &= \sqrt{\left((L_2/L_1)\frac{f(\alpha)}{(1+\alpha)}-1\right)^2+4(L_2/L_1)\frac{f(\alpha)}{(1+\alpha)}}-\left((L_2/L_1)\frac{f(\alpha)}{(1+\alpha)}-1\right)\\ &= (L_2/L_1)\frac{f(\alpha)}{(1+\alpha)}+1-\left((L_2/L_1)\frac{f(\alpha)}{(1+\alpha)}-1\right)=2.\end{aligned}$$

So, we want to prove that

$$\left(\frac{A'}{A}B-B'\right)>\frac{1+L_2/L_1}{(1+\alpha)^2}(1-2b)$$

Some manipulation reveals that

$$\begin{aligned}\frac{A'}{A}B-B' &= (1-b)^2\frac{1+L_2/L_1}{(1+\alpha)^2}+bL_1/L_2\frac{1}{(f(\alpha))^2}\\ &\quad+\frac{b}{(1+\alpha)^2}+b\frac{1+L_1/L_2}{f(\alpha)\,(1+\alpha)}\left(1+\frac{\alpha bL_1/L_2}{f(\alpha)}+b+\frac{\alpha}{(1+\alpha)}b\right)\end{aligned}$$

But it is trivial to establish that this is positive.

Second step: Consider equation $H(x,\alpha_1)=H(x,\alpha_2)$ for any $\alpha_i:\alpha_2>\alpha_1$. It is a linear equation so it has a unique solution. Moreover, so

$$\left(\frac{(L_2/L_1)f(\alpha)}{(1+\alpha)A^2}\right)'_\alpha=L_2/L_1\left(\left(\frac{f(\alpha)}{(1+\alpha)}\right)^{1-2b}\right)'_\alpha.$$

Since $\theta > 1$ (an assumption in EK 2002) $b < 1/2$. That is, $1 - 2b > 0$. This means that $\left(\frac{(L_2/L_1)f(\alpha)}{(1+\alpha)A^2}\right)'_\alpha < 0$ or $-\left(\frac{(L_2/L_1)f(\alpha)}{(1+\alpha)A^2}\right)'_\alpha > 0$. That is, the intercept of $H(x,\alpha)$ with vertical axis is always negative and increasing in α. Thus, $0 > H(0,\alpha_2) > H(0,\alpha_1)$. Since H is U-shaped and $x^0_M(\alpha_2) > x^0_M(\alpha_1) > 0$ (see [2]) then $H(x^0_M(\alpha_1),\alpha_2) < H(x^0_M(\alpha_1),\alpha_1) = 0$.[3] By continuity, there must exist $x^* \in (0, x^0_M(\alpha_1))$ such that $H(x^*,\alpha_1) = H(x^*,\alpha_2)$. Since there is a unique solution to this equation, it follows that $H(x,\alpha_2) < H(x,\alpha_1)$ for all $x \geq x^0_M(\alpha_1)$.

Third step: It is obvious if $J(x,\alpha)$ is fixed and does not change with an increase in alpha, then from the previous two steps we can say that $x_M(\alpha)$ is increasing in alpha. However, with an increase in alpha the curve $J(x,\alpha)$ pivots around some point, with the slope becoming higher or less negative. If we prove that the solution to $J(x,\alpha_2) = J(x,\alpha_1)$ with α_2 just higher than α_1 is less than $x^0_M(\alpha_1)$, then we are done with the proof because the change in $J(x,\alpha)$ amplifies the overall effect on $x_M(\alpha)$. We have

$$J(x,\alpha) = D(\alpha) - F(\alpha)x$$

where

$$\begin{aligned} D(\alpha) &= \left(1 + \frac{\alpha b L_1/L_2}{f(\alpha)}\right)\frac{const}{(1+\alpha)A^2} \\ F(\alpha) &= A(1-b)\frac{const}{(1+\alpha)A^2} \end{aligned}$$

Then,

$$\begin{aligned} J(x,\alpha_2) &= J(x,\alpha_1) \iff \\ x &= \frac{D(\alpha_1) - D(\alpha_2)}{F(\alpha_1) - F(\alpha_2)} \end{aligned}$$

If we take the limit $\alpha_2 \to \alpha_1$, then

$$x = \frac{D'(\alpha)}{F'(\alpha)}$$

Tedious algebra shows that

$$\frac{D'(\alpha)}{F'(\alpha)} = \frac{1}{1-b}\left(\frac{(1+\alpha)}{f(\alpha)}\right)^b \left\{ \left(1 + \frac{\alpha b L_1/L_2}{f(\alpha)}\right) - \frac{(1+\alpha)\left\{\frac{bL_1/L_2}{(f(\alpha))^2} + \left(1 + \frac{\alpha b L_1/L_2}{f(\alpha)}\right)\frac{b(1+L_1/L_2)}{(1+\alpha)f(\alpha)}\right\}}{(1-b)} \right\}$$

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[2] The last inequality comes from $x_M(\alpha) = \frac{-B+\sqrt{B^2+4C}}{2A}$ and noting that $-B + \sqrt{B^2+4C} > -B + \sqrt{B^2} = -B + |B| > -B + B = 0$.

[3] To see this, recall that $x^0_M(\alpha)$ is the highest solution to $H(x,\alpha) = 0$ so that $H_x(x^0_M(\alpha),\alpha) > 0$. Thus, it must be the case that $H(x^0_M(\alpha_1),\alpha_2) < 0$, for otherwise the curve $H(x,\alpha_2)$ would have its lower solution to $H(x,\alpha_2) = 0$ for a level of x higher than $x^0_M(\alpha_1)$ and hence given the U-shape form of H it would follow that $H(0,\alpha_2) > 0$, which is a contradiction.

Next, we compare $\frac{D'(\alpha)}{F'(\alpha)}$ with $x_F(\alpha) = \left(\frac{(1+\alpha)}{f(\alpha)}\right)^b < x_M^0(\alpha)$ (this last inequality follows because $x_F(\alpha) < x_M(\alpha)$ for all Φ_{-m} including $\Phi_{-m} = 0$, but $x_M(\alpha; \Phi_{-m} = 0) = x_M^0(\alpha)$). Algebra shows that this is equivalent to

$$\frac{(1+\alpha)\left\{\frac{L_1/L_2}{(f(\alpha))^2} + \left(1 + \frac{\alpha b L_1/L_2}{f(\alpha)}\right)\frac{(1+L_1/L_2)}{(1+\alpha)f(\alpha)}\right\}}{(1-b)} > \frac{\alpha L_1/L_2}{f(\alpha)} + 1$$

The left side of the inequality positively depends on b. Thus, to prove the inequality we can take $b = 0$, and then simple algebra reveals that the inequality holds. Thus, we proved that the solution of $J(x, \alpha_2) = J(x, \alpha_1)$ for α_2 higher but close to α_1 is strictly less than $x_F(\alpha_1) < x_M^0(\alpha_1)$. **Q.E.D.**