Bubbles and Credit Constraints*

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Abstract

We provide an infinite-horizon model of a production economy with credit-driven stock-price bubbles, in which firms meet stochastic investment opportunities and face credit constraints. Capital is not only an input for production, but also serves as collateral. We show that bubbles on this reproducible asset may arise, which relax collateral constraints and improve investment efficiency. The collapse of bubbles leads to a recession and a stock market crash. We show that there is a credit policy that can eliminate the bubble on firm assets and can achieve the efficient allocation.

Keywords: Credit-driven bubbles, Collateral Constraints, Credit Policy, Asset Price, Arbitrage, Q Theory, Liquidity, Multiple Equilibria

JEL codes: E2, E44

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1 Introduction

This paper provides a theory of credit-driven stock market bubbles. Our theory is motivated by two observations. First, the United States has experienced stock market booms and busts, which may not be explained entirely by fundamentals. Figure 1 presents the monthly real (adjusted for inflation using the Consumer Price Index) Standard and Poor's (S&P) Composite Stock Price Index from January 1871 to January 2011 (upper line), and the corresponding series of real S&P Composite earnings (lower line) for the same period. From this figure, we see that the most dramatic bull market in U.S. history is from July 1982 to August 2000, with the skyrocketing increase in the price index during the late 1990s being the most remarkable. The latter price increase is often attributed to the internet bubble. Yet, the dramatic rise in prices since 1982 is not matched in real earnings growth. The lower line in Figure 1 shows that earnings seem to be oscillating around a slow, steady growth path that has persisted for over a century. Following the peak in 2000, the stock market crashed, reaching the bottom in February 2003. After then the stock market went up and reaching the peak in October 2007. This stock market runup is often attributed to the housing market bubble. Following the burst of the bubble, the U.S. economy has entered the Great Recession, with the stock market drop of 51.7% from October 2007 until March 2009. The recent stock market behavior resembles the runup of the 1920s (the Roaring Twenties), culminating in the 1929 crash. The stock market moves during that period are often used as an example of bubbles and crashes.

Second, some episodes of stock market booms are accompanied by credit booms. This suggests that one possible cause of bubbles is excessive liquidity in the financial system, inducing lax or inappropriate lending standards by the banks.[1] Figure 2 presents the S&P price index in relation to two credit market condition indexes for the United States. Both panels in the figure shows that since 2003 until then, the stock market boom was associated with credit ease and the bust was associated with credit tightening. However, this relationship did not exist during 1990s. This figure suggests that the recent stock market bubble and crash may be credit driven, while it is not the case for the internet bubble and crash.

The above two observations can be applied to many other countries, especially in emerging market countries. For example, overoptimism in 1990s about an "East Asian miracle" gen-

[1] For example, Axel A. Weber, the former president of the Deutsche Bundesbank, has argued that "The past has shown that an overly generous provision of liquidity in global financial markets in connection with a very low level of interest rates promotes the formation of asset-price bubbles." (http://www.bloomberg.com/apps/news?pid=newsarchive&sid=a5S5Boes29lo)

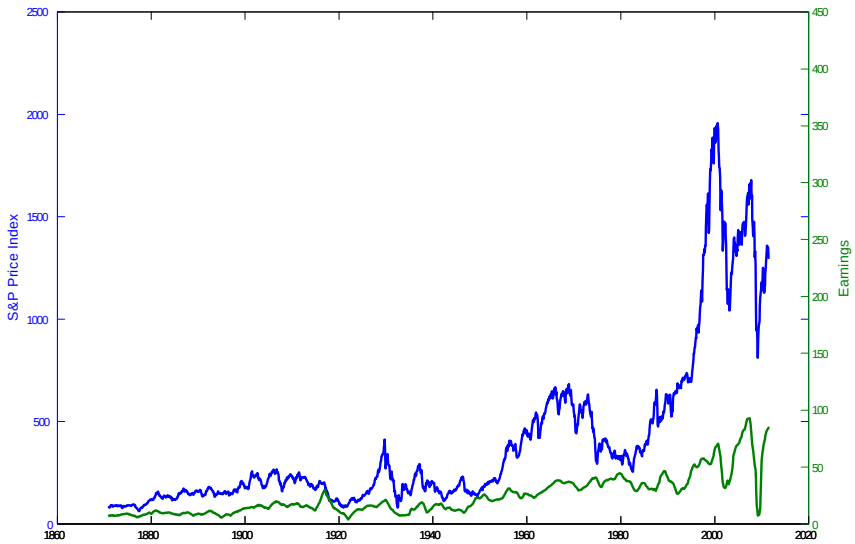

Figure 1: This figures plots the S&P Price Index and earnings. The solid line represents the S&P Price Index and the dashed line represents earnings. The data are downloaded from Robert Shiller's website: http://www.econ.yale.edu/~shiller/data.htm

erated high economic growth in East Asian countries. Capital account and financial market liberalization contributed to large capital inflows and generated a lending boom. The rapid increase in asset prices including housing prices and stock prices were accompanied by a large expansion of domestic credit through under-regulated banking systems. As documented by Collyns and Senhadji (2002), regional stock markets (including Hong Kong, Indonesia, Korea, Malaysia, Philippines, Singapore, Taiwan, and Thailand) trended upward through the first part of the 1990s, generally peaking around 1997 at an average 165% higher than their value in January 1991. Regional stock prices fell sharply after the onset of the crisis in mid-1997 through the end of 1998. In particular, Korea, Malaysia, and Thailand all suffered declines of over 70%.

To formalize our theory, we construct a tractable model of a production economy in which households are infinitely lived and trade firm stocks. We assume that households have linear utility so that the interest rate is equal to the constant subjective discount rate. There is no aggregate uncertainty.[2] A continuum of firms meet idiosyncratic stochastic investment

[2] These two assumptions are adopted for simplicity. Miao and Wang (2011a) introduce a concave utility function to study sectoral bubbles and endogenous growth. Miao and Wang (2011b) study stock market bubbles

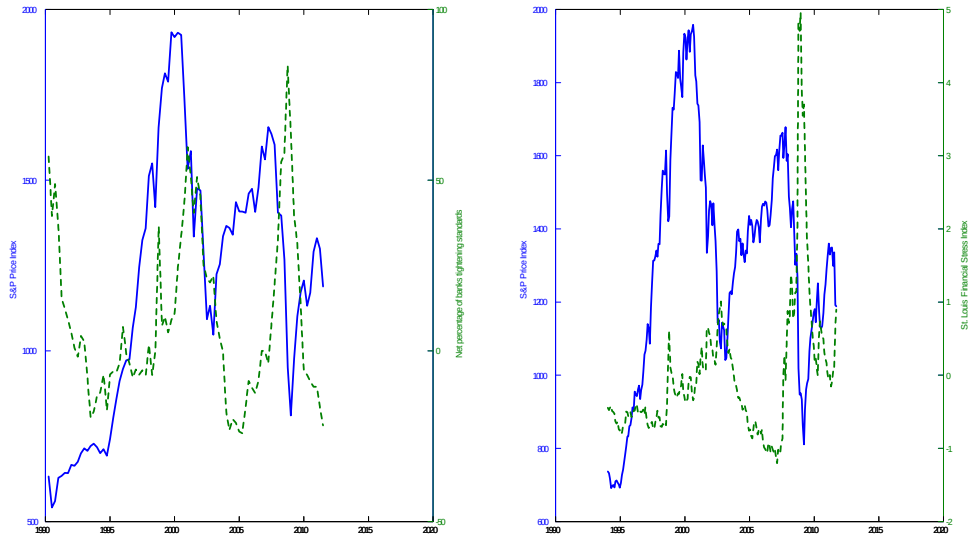

Figure 2: This figure plots S&P Price Index and two credit market condition indexes. The solid lines in the two panels represent the S&P Price Index. The dashed line on the left panel represents Net Percentage of Domestic Respondents Tightening Standards for Commercial and Industrial Loans Large and Medium Firms. The dashed line on the right panel represents the St. Louis Financial Stress Index. The last two series are downloaded from the Federal Reserve Bank of St. Louis.

opportunities as in Kiyotaki and Moore (1997, 2005, 2008) and face credit constraints. We model credit constraints in a way similar to that in Kiyotaki and Moore (1997), Albuquerque and Hopenhayn (2004), and Jermann and Quadrini (2010).[3] Specifically, durable assets (or capital in our model) are used not only as inputs for production, but also as collateral for loans. Borrowing is limited by the market value of the collateral. Unlike Kiyotaki and Moore (1997) who assume that the market value of the collateral is equal to the liquidation value of the collateralized assets, we assume that it is equal to the *going-concern value* of the reorganized firm with these assets. Because the going-concern value is priced in the stock market, it may contain a bubble component. If both lenders and the credit-constrained borrowers (firms in our model) optimistically believe that the collateral value is high possibly because of bubbles, firms will want to borrow more and lenders won't mind lending more. Consequently, firms can finance more investment and accumulate more assets for future production, making their assets indeed more valuable.[4] This positive feedback loop mechanism makes the lenders' and the borrowers' beliefs self-fulfilling and bubbles may sustain in equilibrium. We refer to this equilibrium as the bubbly equilibrium.

Of course, there is another equilibrium in which no one believes in bubbles and hence bubbles do not appear. We call this equilibrium the bubbleless equilibrium. We provide explicit conditions to determine which type of equilibrium can exist. We show that if the degree of pledgeability is sufficiently small, then both bubbleless and bubbly equilibria can exist; otherwise, only the bubbleless equilibrium exists. This result is intuitive. If the degree of pledgeability is sufficiently small, investors have incentives to inflate their asset values to relax the collateral constraint and bubbles may emerge. The emergence of bubbles is accompanied by a credit boom. If the degree of pledgeability is sufficiently large, investors can borrow enough to finance investment. There is no need for them to create bubbles.

We prove that the bubbly equilibrium has two steady states: a bubbly one and a bubbleless one. Both steady states are inefficient due to credit constraints and both are local saddle points. Thus, multiple equilibria in our model are not generated by indeterminacy as in the literature

and business cycles in a DSGE model with risk averse households and aggregate shocks.

[3] We justify the credit constraints in an optimal contract with limited commitment in Section 2.2.

[4] Using firm-level data during the asset price bubble in Japan in the late 1980s, Goyal and Yamada (2004) find that investment responds significantly to stock price bubbles. Using a source of exogenous variation in collateral value provided by the property market collapse in Japan in the early 1990s, Gan (2007) finds a large impact of collateral on the corporate investments of a large sample of Japanese manufacturing firms. She shows that for every 10 percent drop in collateral value, the investment rate of an average firm is reduced by 0.8 percentage point. Chaney, Sraer, and Thesmar (2009) document similar evidence for the US economy during the 1993-2007 period.

surveyed by Benhabib and Farmer (1999) and Farmer (1999). We show that the stable manifold is one dimensional for the bubbly steady state, while it is two dimensional for the bubbleless steady state. On the former stable manifold, bubbles persist in the steady state. But on the latter stable manifold, bubbles eventually burst.[5]

As Tirole (1982) and Santos and Woodford (1997) point out, it is hard to generate rational bubbles for economies with infinitely-lived agents. The intuition is as follows. A necessary condition for bubbles to exist is that the growth rate of bubbles cannot exceed the growth rate of the economy. Otherwise, investors cannot afford to buy into bubbles. In a deterministic economy, bubbles on assets with exogenous payoffs or on intrinsically useless assets must grow at the interest rate by the no-arbitrage principle. Thus, the interest rate cannot exceed the growth rate of the economy. This implies that the present value of aggregate endowments must be infinity. In an overlapping generations economy, this condition implies that the bubbleless equilibrium must be dynamically inefficient (see Tirole (1985)).

In our model, the growth rate of the economy is zero and the interest rate is positive. In addition, the bubbleless equilibrium is dynamically efficient. But how do we reconcile our result with that in Santos and Woodford (1997) or Tirole (1985)? The key is that bubbles in our model are on productive assets with endogenous payoffs. A distinguishing feature of our model is that bubbles on firm assets have real effects and affect the payoffs of these assets. Although a no-arbitrage equation for these bubbles still holds in that the rate of return on bubbles is equal to the interest rate, the growth rate of bubbles is not equal to the interest rate. Rather, it is equal to the interest rate minus the "dividend yield." The dividend yield comes from the fact that bubbles help relax the collateral constraints and allow firms to make more investment. It is equal to the arrival rate of the investment opportunity multiplied by the net benefit of new investment (i.e., Tobin's marginal Q minus 1).

So far, we have only considered deterministic bubbles. Following Blanchard and Watson (1982) and Weil (1987), we construct a third type of equilibrium with stochastic bubbles. In this equilibrium, households believe that there is a positive probability that bubbles will burst at each date. When bubbles burst, they cannot reappear. We show that when all economic agents believe that the probability of bubble bursting is small enough, an equilibrium with stochastic bubbles exists. In contrast to Weil (1987), we show that after a bubble bursts, a

[5] In Chapter 14 of Tirole's (2006) textbook, he shows that there may exist multiple equilibria in a simplified variant of the Kiyotaki and Moore (1997) model. In contrast to our paper, these equilibria are characterized by a one-dimensional nonlinear dynamic system. Some equilibria may exhibit cycles. The steady states of these equilibria are not saddle points. We would like to thank Jean Tirole for a helpful discussion on this point.

recession occurs in that there is a credit crunch and consumption and output fall eventually. In addition, immediately after the bubble bursts, investment falls discontinuously and the stock market crashes, i.e., the stock price also falls discontinuously. Note that the recession and the stock market crash occur without any exogenous shock to the fundamentals of the economy.

What is an appropriate government policy in the wake of a bubble collapse? The inefficiency in our model comes from the firms' credit constraints. The collapse of bubbles tightens these constraints and impairs investment efficiency. To overcome this inefficiency, the government may issue public bonds backed by lump-sum taxes. Both households and firms can trade these bonds, which serve as a store of value to households and firms, and also as collateral to firms. Thus, public assets can relax collateral constraints and play the same role as bubbles do. They deliver dividends to firms, but not to households directly. No arbitrage forces these dividends to zero, making Tobin's marginal Q equal to one. This leads to the efficient capital stock. To support the efficient allocation in equilibrium, the government constantly retires public bonds at the interest rate to maintain a constant total bond value and pays the interest payments of these bonds by levying lump-sum taxes. We show that this policy also completely eliminates the bubbles on firm assets.

Some papers in the literature (e.g., Scheinkman and Weiss (1986), Kocherlakota (1992, 1998), Santos and Woodford (1997) and Hellwig and Lorenzoni (2009)) also find that infinite-horizon models with borrowing constraints may generate rational bubbles. Unlike these papers which study pure exchange economies, our paper analyzes a production economy. As mentioned above, our paper differs from these and the papers cited below in that we focus on bubbles in stock prices whose payoffs are endogenously determined by investment and affected by bubbles.[6] In addition, we focus on borrowing constraints on the firm side instead of the household side.

Our paper is closely related to Caballero and Krishnamurthy (2006), Kocherlakota (2009), Wang and Wen (2009), Farhi and Tirole (2010), and Martin and Ventura (2010a,b). Like our paper, these papers contain the idea that bubbles can help relax borrowing constraints and improve investment efficiency. Building on Kiyotaki and Moore (2008), Kocherlakota (2009) studies an economy with infinitely lived entrepreneurs. Entrepreneurs meet stochastic investment opportunities and are subject to collateral constraints. Land is used as the collateral. Unlike Kiyotaki and Moore (1997) or our paper, Kocherlakota (2009) assumes that land is intrinsically useless (i.e. it has no rents or dividends) and cannot be used as an input for

[6] See Scheinkman and Xiong (2003) and Burnside, Eichenbaum and Rebelo (2011) for models of bubbles based on heterogeneous beliefs. See Shiller (2005) for a theory of bubbles based on irrational exuberance. See Brunnermeier (2009) for a survey of various theories of bubbles.

production. Wang and Wen (2011) provide a model similar to that in Kocherlakota (2009). They study asset price volatility and bubbles that may grow on assets with exogenous rents. They assume that these assets cannot be used as an input for production. Our model can also generate bubbles on intrinsically useless assets as long as these assets can be used to finance investment and households face short sales constraints. These assumptions are standard in the literature (e.g., Kocherlakota (2009) and Wang and Wen (2011)).

Building on Diamond (1965) and Tirole (1985), Caballero and Krishnamurthy (2006), Farhi and Tirole (2010), and Martin and Ventura (2010a,b) study bubbles in overlapping generations models with credit constraints. Caballero and Krishnamurthy (2006) show that stochastic bubbles are beneficial because they provide domestic stores of value, thereby reducing capital outflows while increasing investment. But they come at a cost, as they expose the country to bubble crashes and capital flow reversals. Farhi and Tirole (2010) assume that entrepreneurs may use bubbles and outside liquidity to relax the credit constraints. They study the interplay between inside and outside liquidity. Martin and Ventura (2010b) use a model with bubbles to shed light on the recent financial crisis. Unlike our paper, all these papers show that the growth rate of bubbles is equal to the interest rate because they study bubbles on intrinsically useless assets.

Our discussion of credit policy is related to Caballero and Krishnamurthy (2006) and Kocherlakota (2009). As in their studies, government bonds can serve as collateral to relax credit constraints in our model. Unlike their proposed policies, our proposed policy requires that government bonds be backed by lump-sum taxes and it can make the economy achieve the efficient allocation. Unbacked public assets are intrinsically useless and may have positive value (a bubble) if households face short sales constraints. Issuing unbacked public assets can boost the economy after the collapse of stock-price bubbles. But the real allocation is still inefficient and the bubble on unbacked public assets can burst. After bursting, the economy enters a recession again.

The rest of the paper is organized as follows. Section 2 presents the model. Section 3 derives the equilibrium system. Section 4 analyzes the bubbleless equilibrium, while Section 5 analyzes the bubbly equilibrium. Section 6 studies stochastic bubbles. Section 7 introduces public assets and studies government credit policy. Section 8 concludes. Appendix A contains all proofs. Appendices B-D consider several variations and analyze the robustness of our results. Specifically, Appendix B introduces capacity utilization and analyzes stochastic bubbles. Appendix C studies a discrete-time setup where firms can borrow and save intertemporally.

Appendix D analyzes a discrete-time setup when firms face idiosyncratic investment-specific shocks with a continuous distribution.

2 The Baseline Model

We consider an infinite-horizon economy. There is no aggregate uncertainty. Time is denoted by $t = 0$, dt, $2dt$, $3dt$, ... The length of a time period is dt. For analytical convenience, we shall take the limit of this discrete-time economy as dt goes to zero when characterizing equilibrium dynamics. The continuous-time model is more convenient for analyzing local dynamics around a steady state. Instead of presenting the continuous-time model directly, we start with the model in discrete time in order to make the intuition transparent.

2.1 Households

There is a continuum of identical households of unit mass. Each household is risk neutral and derives utility from a consumption stream $\{C_t\}$ according to the following utility function:

$$\sum_{t\in\{0,dt,2dt,\ldots\}} e^{-rt} C_t dt,$$

where r is the subjective rate of time preference.[7] Households supply labor inelastically. The labor supply is normalized to one. Households trade firm stocks and risk-free household bonds. The net supply of household bonds is zero and the net supply of any stock is one. Because there is no aggregate uncertainty, r is equal to the risk-free rate (or interest rate) and also equal to the rate of the return for each stock.

2.2 Firms

There is a continuum of firms of unit mass. Firms are indexed by $j \in [0,1]$. Each firm j combines labor N_t^j and capital K_t^j to produce output according to the following Cobb-Douglas production function:

$$Y_t^j = (K_t^j)^\alpha (N_t^j)^{1-\alpha}, \quad \alpha \in (0,1).$$

After solving the static labor choice problem, we obtain the operating profits

$$R_t K_t^j = \max_{N_t^j} (K_t^j)^\alpha (N_t^j)^{1-\alpha} - w_t N_t^j, \tag{1}$$

[7] Introducing a general concave utility function allows us to endogenize interest rate, but it makes analysis more complex. It will not change our key insights (see Miao and Wang (2011a,b)).

where w_t is the wage rate and

$$R_t = \alpha \left(\frac{w_t}{1-\alpha} \right)^{\frac{\alpha-1}{\alpha}}. \tag{2}$$

We will show later that R_t is equal to the marginal product of capital or the rental rate of capital.

Following Kiyotaki and Moore (1997, 2005, 2008), we assume that each firm j meets an opportunity to make investment in capital with probability πdt in period t. With probability $1 - \pi dt$, no investment opportunity arrives. Thus, capital evolves according to:

$$K_{t+dt}^j = \begin{cases} (1-\delta dt) K_t^j + I_t^j & \text{with probability } \pi dt \\ (1-\delta dt) K_t^j & \text{with probability } 1-\pi dt \end{cases}, \tag{3}$$

where $\delta > 0$ is the depreciation rate of capital and I_t^j is the investment level. This assumption captures firm-level investment lumpiness and generates ex post firm heterogeneity. Assume that the arrival of the investment opportunity is independent across firms and over time. In Appendix D, we study the case where firms are subject to idiosyncratic investment-specific shocks with a continuous distribution. This alternative modeling does not change our key insights. In a model without investment-specific shocks, Miao and Wang (2012) introduce idiosyncratic productivity shocks and show that credit-driven bubbles can still emerge. These bubbles help raise total factor productivity endogenously.

Let the ex ante firm value (or stock value) prior to observing the arrival of investment opportunities be $V_t(K_t^j)$, where we suppress aggregate state variables in the argument. It satisfies the following Bellman equation:

$$\begin{aligned} V_t(K_t^j) \quad = \quad & \max_{I_t^j} \; R_t K_t^j dt - \pi I_t^j dt + e^{-rdt} V_{t+dt}((1-\delta dt) K_t^j + I_t^j)\pi dt \\ & + e^{-rdt} V_{t+dt}((1-\delta dt) K_t^j)(1-\pi dt), \end{aligned} \tag{4}$$

subject to some constraints on investment to be specified next. As will be shown in Section 3, the optimization problem in (4) is not well defined if there is no constraint on investment given our assumption of the constant returns to scale technology. Thus, we impose some upper bound and lower bound on investment. For the lower bound, we assume that investment is irreversible in that $I_t^j \geq 0$. It turns out this constraint will never bind in our analysis below. For the upper bound, we assume that investment is financed by internal funds and external borrowing. We also assume that external equity is so costly that no firms would raise new

equity to finance investment.[8]

We now write the investment constraint as:

$$0 \leq I_t^j \leq R_t K_t^j + L_t^j, \tag{5}$$

where $R_t K_t^j$ represents internal funds and L_t^j represents loans from financial intermediaries. To reduce the number of state variables and keep the model tractable, we consider intratemporal loans as in Carlstrom and Fuerst (1997) and Jermann and Quadrini (2010). These loans are taken at the beginning of the period and repaid at the end of the period. They do not have interests. In Appendix C, we incorporate intertemporal bonds with interest payments and allow firms to save. We show that our key insights and analysis carry over to this setup.

The key assumption of our model is that loans are subject to the collateral constraint:

$$L_t^j \leq e^{-rdt} V_{t+dt}(\xi K_t^j). \tag{6}$$

The motivation of this constraint is similar to that in Kiyotaki and Moore (1997): Firm j pledges a fraction $\xi \in (0, 1]$ of its assets (capital stock) K_t^j at the beginning of period t as the collateral. The parameter ξ may represent the degree of pledgeability or the extent of financial market imperfections. It is the key parameter for our analysis below. At the end of period t, the stock market value of the collateral is equal to $e^{-rdt} V_{t+dt}(\xi K_t^j)$. The lender never allows the loan repayment L_t^j to exceed this value. If this condition is violated, then firm j may take loans L_t^j and walk away, leaving the collateralized asset ξK_t^j behind. In this case, the lender runs the firm with the collateralized assets ξK_t^j at the beginning of period $t + dt$ and obtains the smaller firm value $e^{-rdt} V_{t+dt}(\xi K_t^j)$ at the end of period t, which is the collateral value. Alternatively, the lender sells the collateralized assets to a third party at the value $e^{-rdt} V_{t+dt}(\xi K_t^j)$. The third party runs the firm with these assets and obtains the *going-concern value* $e^{-rdt} V_{t+dt}(\xi K_t^j)$. Unlike Kiyotaki and Moore (1997), we have implicitly assumed that firm assets are not specific to a particular owner. Any owner can operate the assets using the same technology. Thus, unlike in Kiyotaki and Moore (1997), the collateral value in our model is not equal to the the liquidation value.

[8] This assumption reflects the fact that external equity financing is more costly than debt financing. Bernanke et al. (1999), Carsltrom and Fuerst (1997), and Kiyotaki and Moore (1997) make the same assumption. We can relax this assumption by allowing firms to raise a limited amount of new equity so that we can rewrite (5) as

$$I_t^j \leq R_t K_t^j + a K_t^j + L_t^j,$$

where $a K_t^j$ represents the upper bound of new equity. In this case, our analysis and insights still hold with small modification. Also see Miao and Wang (2010) for a model where firms can endogenously choose the debt-equity mix.

We may interpret the collateral constraint in (6) as an incentive constraint in an optimal contract between firm j and the lender with limited commitment:[9] Given a history of information at date t, in the time interval $t + dt$, the contract specifies investments I_t^j and loans L_t^j at the beginning of period t, and repayments L_t^j at the end of period t, only when an investment opportunity arrives with Poisson probability πdt. When no investment opportunity arrives, the firm does not invest and hence does not borrow. Firm j may default on debt at the end of period t. If it happens, then the firm and the lender renegotiate the loan repayment. In addition, the lender reorganizes the firm. Because of default costs, the lender can only seize a fraction ξ of capital K_t^j. Alternatively, we may interpret ξ as an efficiency parameter in that the lender may not be able to efficiently use the firm's assets K_t^j. The lender can run the firm with these assets at the beginning of period $t + dt$ and obtains firm value $e^{-rdt}V_{t+dt}(\xi K_t^j)$. Or it can sell these assets to a third party at the going-concern value $e^{-rdt}V_{t+dt}(\xi K_t^j)$ if the third party can run the firm using assets ξK_t^j at the beginning of period $t + dt$. This value is the threat value (or the collateral value) to the lender at the end of period t. Following Jermann and Quadrini (2010), we assume that the firm has all the bargaining power in the renegotiation and the lender gets only the threat value. The key difference between our modeling and that of Jermann and Quadrini (2010) is that the threat value to the lender is the going concern value in our model, while Jermann and Quadrini (2010) assume that the lender liquidates the firm's assets and obtains the liquidation value in the even of default.[10]

Enforcement requires that, when the investment opportunity arrives at date t, the continuation value to the firm of not defaulting is not smaller than the continuation value of defaulting, that is,

$$\begin{aligned} & e^{-rdt}V_{t+dt}((1-\delta dt)\,K_t^j + I_t^j) - L_t^j \\ \geq\ & e^{-rdt}V_{t+dt}((1-\delta dt)\,K_t^j + I_t^j) - e^{-rdt}V_{t+dt}(\xi K_t^j). \end{aligned} \tag{7}$$

This incentive constraint is equivalent to the collateral constraint in (6). This constraint ensures that there is no default in an optimal contract.

In the continuous-time limit, the collateral constraint becomes

$$L_t^j \leq V_t(\xi K_t^j). \tag{8}$$

[9] See Albuquerque and Hopenhayn (2004) and Alvarez and Jermann (2000) for related contracting problems.

[10] U.S. Bankruptcy law has recognized the need to preserve going concern value when reorganizing businesses in order to maximize recoveries by creditors and shareholders (see 11 U.S.C. 1101 et seq.). Bankruptcy laws seek to preserve going concern value whenever possible by promoting the reorganization, as opposed to the liquidation, of businesses.

Note that our modeling of collateral constraint is different from that of Kiyotaki and Moore (1997). We may write the Kiyotaki-Moore-type collateral constraint in our continuous-time framework as:

$$L_t^j \leq \xi Q_t K_t^j, \tag{9}$$

where Q_t represents the shadow price of capital. The expression $\xi Q_t K_t^j$ is the shadow value of the collateralized assets or the liquidation value.[11] In Section 5, we shall argue that this type of collateral constraint will rule out bubbles. By contrast, according to (6), we allow the collateralized assets are valued in the stock market as the going-concern value when the firm is reorganized and keeps running using the collateralized assets after default. If both the firm and the lender believe that the firm's assets may be overvalued due to stock market bubbles, then these bubbles will relax the collateral constraint, which provides a positive feedback loop mechanism.

2.3 Competitive Equilibrium

Let $K_t = \int_0^1 K_t^j dj$, $I_t = \int_0^1 I_t^j dj$, $N_t = \int_0^1 N_t^j dj$, and $Y_t = \int_0^1 Y_t^j dj$ be the aggregate capital stock, the aggregate investment, the aggregate labor demand, and aggregate output. Then a competitive equilibrium is defined as sequences of $\{Y_t\}$, $\{C_t\}$, $\{K_t\}$, $\{I_t\}$, $\{N_t\}$, $\{w_t\}$, $\{R_t\}$, $\{V_t(K_t^j)\}$, $\{I_t^j\}$, $\{K_t^j\}$, $\{N_t^j\}$ and $\{L_t^j\}$ such that households and firms optimize and markets clear in that:

$$\begin{aligned} N_t &= 1, \\ C_t + \pi I_t &= Y_t, \\ K_{t+dt} &= (1 - \delta dt) K_t + I_t \pi dt. \end{aligned}$$

3 Equilibrium System

We first solve an individual firm's optimal contract problem (4) subject to (3), (5), and (6) when the wage rate w_t or the rental rate R_t in (2) is taken as given. This problem does not give a contraction mapping and hence may admit multiple solutions. We conjecture that ex

[11] Note that our model differs from the Kiyotaki and Moore model in market arrangements, besides other specific modeling details. Kiyotaki and Moore assume that there is a market for physical capital (corresponding to land in their model), but there is no stock market for trading firm shares. In addition, they assume that households and entrepreneurs own firms and trade physical capital in the capital market. By contrast, we assume that households trade firm shares in the stock market and that firms own physical captial and make investment

ante firm value takes the following form:

$$V_t(K_t^j) = v_t K_t^j + b_t, \tag{10}$$

where v_t and b_t are to be determined and depend on aggregate states only. Note that $b_t = 0$ is a possible solution. In this case, we may interpret $v_t K_t^j$ as the fundamental value of the firm. The fundamental value is proportional to the firm's assets K_t^j, which has the same form as that derived in Hayashi (1982). The firm has no fundamental value if it has no assets ($K_t^j = 0$). There may be another solution in which $b_t > 0$. In this case, we interpret b_t as a bubble.[12]

Let Q_t be the Lagrange multiplier associated with the constraint (3) if the investment opportunity arrives. It represents the shadow price of capital or Tobin's marginal Q. The following result characterizes firm j's optimization problem:

Proposition 1 *Suppose $Q_t > 1$ and let w_t be given. Then the optimal investment level when the investment opportunity arrives is given by:*

$$I_t^j = R_t K_t^j + \xi Q_t K_t^j + B_t, \tag{11}$$

where R_t is given by (2) and

$$B_t = e^{-rdt} b_{t+dt}, \tag{12}$$

$$Q_t = e^{-rdt} v_{t+dt}. \tag{13}$$

In addition,

$$v_t = R_t dt + (1 - \delta dt) Q_t + (Q_t - 1)(R_t + \xi Q_t) \pi dt, \tag{14}$$

$$b_t = B_t + (Q_t - 1) B_t \pi dt. \tag{15}$$

and the transversality condition holds:

$$\lim_{T\to\infty} e^{-rTdt} Q_T K_{T+dt}^j = 0, \quad \lim_{T\to\infty} e^{-rTdt} b_T = 0.$$

The intuition behind this proposition is as follows. When the investment opportunity arrives, an additional unit of investment costs the firm one unit of the consumption good, but generates an additional value of Q_t, where Q_t satisfies (13). This equation and equation (10) reveal that

$$Q_t = e^{-rdt} \frac{\partial V_{t+dt}(K_{t+dt})}{\partial K_{t+dt}}.$$

[12] We can solve for ex post firm value after the realization of idiosyncratic shocks. Ex post firm value can also contain a bubble. This bubble depends on the realization of idiosyncratic shocks. See Appendix B-D for related analysis in various setups.

Thus, Q_t represents the marginal value of the firm following a unit increase in capital at time $t + dt$ in time-t dollars, i.e., Tobin's marginal Q. If $Q_t > 1$, the firm will make the maximal possible level of investment. If $Q_t = 1$, the investment level is indeterminate. If $Q_t < 1$, the firm will make the minimal possible level of investment. This investment choice is similar to Tobin's Q theory (Tobin (1969) and Hayashi (1982)). In what follows, we impose assumptions to ensure $Q_t > 1$ at least in the neighborhood of the steady state equilibrium. We thus obtain the investment rule given in (11). Substituting this rule and equation (10) into the Bellman equation (4) and matching coefficients, we obtain equations (14) and (15).

More specifically, we rewrite the firm's problem explicitly as:

$$\begin{aligned} v_t K_t^j + b_t &= \max_{I_t^j} \; R_t K_t^j dt - \pi I_t^j dt + \underbrace{e^{-rdt} v_{t+dt}}_{Q_t} \pi I_t^j dt \\ &\quad + \underbrace{e^{-rdt} v_{t+dt}}_{Q_t} (1 - \delta dt) K_t^j + \underbrace{e^{-rdt} b_{t+dt}}_{B_t}, \end{aligned}$$

subject to

$$I_t^j \leq R_t K_t^j + e^{-rdt} V_{t+dt}(\xi K_t^j) = R_t K_t^j + \underbrace{e^{-rdt} v_{t+dt}}_{Q_t} \xi K_t^j + \underbrace{e^{-rdt} b_{t+dt}}_{B_t}.$$

The existence of a bubble $b_t > 0$ on the collateralized assets allows the borrowing constraint to be relaxed. Thus, bubbles are accompanied by a credit boom, leading the firm to make more investments. This raises firm value and supports the inflated market value of assets. This positive feedback loop mechanism generates a stock-price bubble.

Although our model features a constant-returns-to-scale technology, marginal Q is not equal to average Q in the presence of bubbles, because average Q is equal to

$$\frac{e^{-rdt} V_{t+dt}(K_{t+dt})}{K_{t+dt}} = Q_t + \frac{B_t}{K_{t+dt}}, \text{ for } B_t \neq 0.$$

Thus, the existence of stock price bubbles invalidates Hayashi's (1982) result. In the empirical investment literature, researchers typically use average Q to replace marginal Q under the constant returns to scale assumption because marginal Q is not observable. Our analysis demonstrates that the existence of collateral constraints implies that stock prices may contain a bubble component that makes marginal Q not equal to average Q.

Next, we aggregate individual firm's decision rules and impose market-clearing conditions. We then characterize a competitive equilibrium by a system of nonlinear difference equations:

Proposition 2 *Suppose $Q_t > 1$. Then the equilibrium sequences (B_t, Q_t, K_t), for $t = 0$, dt, $2dt, ...$, satisfy the following system of nonlinear difference equations:*

$$B_t = e^{-rdt} B_{t+dt}[1 + \pi(Q_{t+dt} - 1)dt], \tag{16}$$

$$Q_t = e^{-rdt} \left[R_{t+dt}dt + (1 - \delta dt)Q_{t+dt} + (R_{t+dt} + \xi Q_{t+dt})(Q_{t+dt} - 1)\pi dt\right], \tag{17}$$

$$K_{t+dt} = (1 - \delta dt)K_t + \pi (R_t K_t + \xi Q_t K_t + B_t) dt, \ K_0 \text{ given}, \tag{18}$$

and the transversality condition:

$$\lim_{T\to\infty} e^{-rTdt} Q_T K_{T+dt} = 0, \quad \lim_{T\to\infty} e^{-rTdt} B_T = 0,$$

where $R_t = \alpha K_t^{\alpha-1}$.

When $dt = 1$, the above system reduces to the usual discrete-time characterization of equilibrium. However, this system is not convenient for analytically characterizing local dynamics. We may solve this system numerically by assigning parameter values. Instead of pursuing this route, we use analytical methods in the continuous-time limit as dt goes to zero. To compute the limit, we use the heuristic rule $dX_t = X_{t+dt} - X_t$ for any variable X_t. We also use the notation $\dot{X}_t = dX_t/dt$. We obtain the following:

Proposition 3 *Suppose $Q_t > 1$. Then in the continuous-time limit as $dt \to 0$, the equilibrium dynamics (B_t, Q_t, K_t) satisfy the following system of differential equations:*

$$\dot{B}_t = rB_t - B_t\pi(Q_t - 1), \tag{19}$$

$$\dot{Q}_t = (r + \delta) Q_t - R_t - \pi(R_t + \xi Q_t)(Q_t - 1), \tag{20}$$

$$\dot{K}_t = -\delta K_t + \pi(R_t K_t + \xi Q_t K_t + B_t), \ K_0 \text{ given}, \tag{21}$$

and the transversality condition:

$$\lim_{T\to\infty} e^{-rT} Q_T K_T = 0, \quad \lim_{T\to\infty} e^{-rT} B_T = 0,$$

where $R_t = \alpha K_t^{\alpha-1}$. In addition, $Q_t = v_t$ and $B_t = b_t$ so that the market value of firm j is given by $V_t(K_t^j) = Q_t K_t^j + B_t$.

After obtaining the solution for (B_t, Q_t, K_t), we can derive the equilibrium wage rate $w_t = (1 - \alpha) K_t^\alpha$, the rental rate $R_t = \alpha K_t^{\alpha-1}$, aggregate output $Y_t = K_t^\alpha$, aggregate investment,

$$I_t = R_t K_t + \xi Q_t K_t + B_t, \tag{22}$$

and aggregate consumption $C_t = Y_t - \pi I_t$. We focus on two types of equilibrium.[13] The first type is bubbleless, for which $B_t = 0$ for all t. In this case, the market value of firm j is equal to its fundamental value in that $V_t(K_t^j) = Q_t K_t^j$. The second type is bubbly, for which $B_t > 0$ for some t. We assume that assets can be freely disposed of so that the bubbles B_t cannot be negative. In this case, firm value contains a bubble component in that $V_t(K_t^j) = Q_t K_t^j + B_t$ with $B_t > 0$. We next study these two types of equilibrium.

4 Bubbleless Equilibrium

In a bubbleless equilibrium, $B_t = 0$ for all t. Equation (19) becomes an identity. We only need to focus on (Q_t, K_t) determined by the differential equations (20) and (21) in which $B_t = 0$ for all t. In the continuous time limit, $v_t = Q_t$

We first analyze the steady state. In the steady state, all aggregate variables are constant over time so that $\dot{Q}_t = \dot{K}_t = 0$. We use X to denote the steady state value of any variable X_t. By (20) and (21), we obtain the following steady-state equations:

$$0 = (r + \delta) Q - R - \pi(R + \xi Q)(Q - 1), \tag{23}$$

$$0 = -\delta K + \pi(RK + \xi QK). \tag{24}$$

We use a variable with an asterisk to denote its value in the bubbleless equilibrium. Solving equations (23)-(24) yields:

Proposition 4 *(i) If*

$$\xi \geq \frac{\delta(1-\pi)}{\pi} - r, \tag{25}$$

then there exists a unique bubbleless steady state equilibrium with $Q_t^* = Q_E \equiv 1$ and $K_t^* = K_E$, where K_E is the efficient capital stock satisfying $\alpha(K_E)^{\alpha-1} = r + \delta$.

(ii) If

$$0 < \xi < \frac{\delta(1-\pi)}{\pi} - r, \tag{26}$$

then there exists a unique bubbleless steady-state equilibrium with

$$Q^* = \frac{\delta(1-\pi)}{\pi} \frac{1}{r+\xi} > 1, \tag{27}$$

$$\alpha (K^*)^{\alpha-1} = \frac{\delta(1-\pi)}{\pi} \frac{r}{r+\xi} + \delta. \tag{28}$$

[13] We focus on the case where either all firms have bubbles in their stock prices or no firms have bubbles in their stock prices. It is possible to have another type of equilibrium in which only a fraction of firms have bubbles in their stock prices.

In addition, $K^* < K_E$.

Assumption (25) says that if firms pledge sufficient assets as the collateral, then the collateral constraints will not bind in equilibrium. The competitive equilibrium allocation is the same as the efficient allocation. The efficient allocation is achieved by solving a social planner's problem in which the social planner maximizes the representative household's utility subject to the resource constraint only. Note that we assume that the social planner also faces stochastic investment opportunities, like firms in a competitive equilibrium. Thus, one may view our definition of the efficient allocation as the constrained efficient allocation. Unlike firms in a competitive equilibrium, the social planner is not subject to collateral constraints.

Assumption (26) says that if firms do not pledge sufficient assets as the collateral, then the collateral constraints will be sufficiently tight so that firms are credit constrained in the neighborhood of the steady-state equilibrium in which $Q^* > 1$. We can then apply Proposition 3 in this neighborhood. Proposition 4 also shows that the steady-state capital stock for the bubbleless competitive equilibrium is less than the efficient steady-state capital stock. This reflects the fact that not enough resources are transferred from savers to investors due to the collateral constraints.

Note that for (26) to hold, the arrival rate π of the investment opportunity must be sufficiently small, holding everything else constant. The intuition is that if π is too high, then too many firms will have investment opportunities so that the accumulated aggregate capital stock will be large, thereby lowering the capital price Q to the efficient level as shown in part (i) of Proposition 4. In this case, firms can accumulate sufficient internal funds and do not need external financing. Thus, the collateral constraints will not bind and the economy will reach the first best. Condition (26) requires that technological constraints at the firm level be sufficiently tight.

Now, we study the stability of the steady state and the dynamics of the equilibrium system. We use the phase diagram in Figure 3 to describe the two-dimensional dynamic system for (Q_t, K_t). It is straightforward to show that the $\dot{K}_t = 0$ locus is upward sloping. Above this line, $\dot{K}_t < 0$, and below this line $\dot{K}_t > 0$. Turn to the $\dot{Q}_t = 0$ locus. One can verify that on the $\dot{Q}_t = 0$ locus, $dK/dQ|_{Q\to 1} < 0$ and $dK/dQ|_{Q\to\infty} > 0$. But for general values of $Q > 1$, we cannot determine the sign of dK/dQ. Above the $\dot{Q}_t = 0$ line, $\dot{Q}_t > 0$, and below the $\dot{Q}_t = 0$ line, $\dot{Q}_t < 0$. In addition, the $\dot{Q}_t = 0$ line and the $\dot{K}_t = 0$ line intersect only once at the steady state (Q^*, K^*). The slope of the $\dot{K}_t = 0$ line is always larger than that of the $\dot{Q}_t = 0$ line.

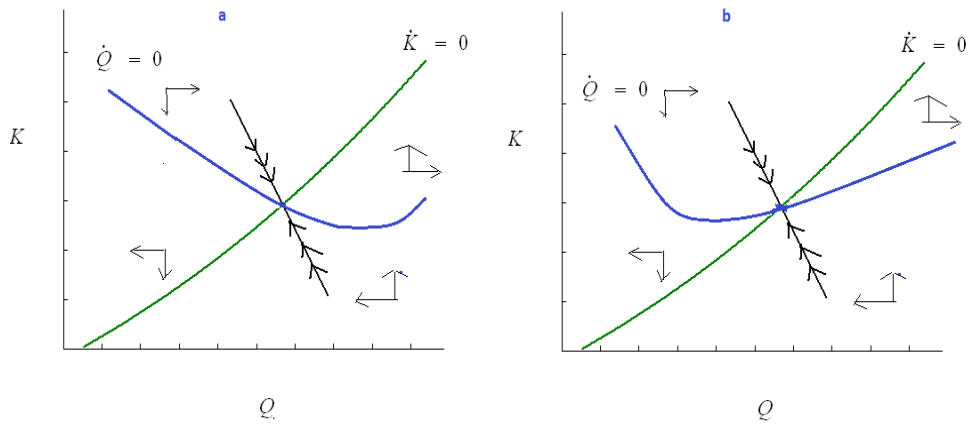

Figure 3: Phase diagram for the dynamics of the bubbleless equilibrium.

For $Q < Q^*$, the $\dot{Q}_t = 0$ line is above the $\dot{K}_t = 0$ line. For $Q > Q^*$, the opposite is true. In summary, two cases exist as illustrated in Figure 3. For both cases, there is a unique saddle path such that for any given initial value K_0, when Q_0 is on the saddle path, the economy approaches the long-run steady state.

5 Bubbly Equilibrium

In this section, we study the bubbly equilibrium in which $B_t > 0$ for some t. We shall analyze the dynamic system for (B_t, Q_t, K_t) given in (19)-(21). Before we conduct a formal analysis later, we first explain why bubbles can exist in our model. The key is to understand equation (19), rewritten as:

$$\frac{\dot{B}_t}{B_t} + \pi(Q_t - 1) = r, \text{ for } B_t \neq 0. \tag{29}$$

The first term on the left-hand side is the rate of capital gains of bubbles. The second term represents "dividend yields", as we will explain below. Thus, equation (19) or (29) reflects a no-arbitrage relation in that the rate of return on bubbles must be equal to the interest rate. A similar relation also appears in the literature on rational bubbles, e.g., Blanchard and Watson (1982), Tirole (1985), Weil (1987, 1993), and Farhi and Tirole (2010). This literature typically studies bubbles on zero-payoff assets or unproductive assets with exogenously given payoffs. In this case, the second term on the left-hand side of (29) vanishes and bubbles grow at the

rate of interest. If we adopt collateral constraint (8) as in Kiyotaki and Moore (1997), then we can also show that bubbles grow at the rate of interest. In an infinite-horizon economy, the transversality condition rules out these bubbles. In an overlapping generation economy, for bubbles to exist, the interest rate must be less than the growth rate of the economy in the bubbleless equilibrium. This means that the bubbleless equilibrium must be dynamically inefficient (see Tirole (1985)).

Unlike this literature, bubbles in our model are on reproducible real assets and also influence their fundamentals (or dividends). Specifically, each unit of the bubble raises the collateral value by one unit and hence allows the firm to borrow an additional unit. The firm then makes one more unit of investment when an investment opportunity arrives. This unit of investment raises firm value by Q_t. Subtracting one unit of costs, we then deduce that the second term on the left-hand side of (29) represents the net increase in firm value for each unit of bubbles. This is why we call this term dividend yields. Dividend payouts make the growth rate of bubbles less than the interest rate. Thus, the transversality condition cannot rule out bubbles in our model. We can also show that the bubbleless equilibrium is dynamically efficient in our model. Specifically, the golden rule capital stock is given by $K_{GR} = (\delta/\alpha)^{\frac{1}{\alpha-1}}$. One can verify that $K^* < K_{GR}$. Thus, one cannot use the condition for the overlapping generations economies in Tirole (1985) to ensure the existence of bubbles. Below we will give new conditions to ensure the existence of bubbles in our model.

5.1 Steady State

We first study the existence of a bubbly steady state in which $B > 0$. We use a variable with a subscript b to denote this variable's bubbly steady state value. By Proposition 3, (B, Q_b, K_b) satisfies equations (23) and

$$0 = rB - B\pi(Q - 1), \tag{30}$$

$$0 = -\delta K + [RK + \xi QK + B]\pi. \tag{31}$$

Using these equations, we can derive:

Proposition 5 *There exists a bubbly steady state satisfying*

$$\frac{B}{K_b} = \frac{\delta}{\pi} - \frac{r + \delta + \xi}{1 + r}\frac{r + \pi}{\pi} > 0, \tag{32}$$

$$Q_b = \frac{r}{\pi} + 1 > 1, \tag{33}$$

$$\alpha (K_b)^{\alpha-1} = \frac{(1-\xi)r+\delta}{1+r}\left(\frac{r}{\pi}+1\right), \tag{34}$$

if and only if the following condition holds:

$$0 < \xi < \frac{\delta(1-\pi)}{r+\pi} - r. \tag{35}$$

In addition, (i) $Q_b < Q^*$*, (ii)* $K_{GR} > K_E > K_b > K^*$*, and (iii) the bubble-asset ratio* B/K_b *decreases with* ξ*.*

From equations (23), (30) and (31), we can immediately derive (32)-(34). We can then immediately see that condition (35) is equivalent to $B/K_b > 0$. This condition reveals that bubbles occur when ξ is sufficiently small or the collateral constraint is sufficiently tight.[14] The intuition is the following. When the collateral constraint is too tight, firms prefer to overvalue their assets in order to raise their collateral value. In this way, they can borrow more and invest more. As a result, bubbles may emerge. If the collateral constraint is not tight enough, firms can borrow sufficient funds to finance investment. They have no incentive to create a bubble.

Note that condition (35) implies condition (26). Thus, if condition (35) holds, then there exist two steady state equilibria: one is bubbleless and the other is bubbly. The bubbleless steady state is analyzed in Proposition 4. Propositions 5 and 4 reveal that the steady-state capital price is lower in the bubbly equilibrium than in the bubbleless equilibrium, i.e., $Q_b < Q^*$. The intuition is as follows. In a bubbleless or a bubbly steady state, the investment rate must be equal to the rate of capital depreciation such that the capital stock is constant over time (see equations (24) and (31)). Bubbles relax collateral constraints and induce firms to make more investment, compared to the case without bubbles. To maintain the same steady-state investment rate, the capital price in the bubbly steady state must be lower than that in the bubbleless steady state.

Do bubbles crowd out capital in the steady state? In Tirole's (1985) overlapping generations model, households may use part of savings to buy bubble assets instead of accumulating capital. Thus, bubbles crowd out capital in the steady state. In our model, bubbles are on reproducible assets. If the capital price were the same for both bubbly and bubbleless steady states, then bubbles would induce firms to invest more and hence to accumulate more capital stock. However, there is a general equilibrium price feedback effect as discussed earlier. The lower capital price in the bubbly steady state discourages firms to accumulate more capital

[14] In Appendix C, we show that bubbles in stock prices can exist even for $\xi = 1$ when firms face idiosyncratic investment-specific shocks with a continuous distribution. In this case, the collateral constraint is still too tight.

stock. The net effect is that bubbles lead to higher capital accumulation, unlike Tirole's (1985) result. Note that bubbles do not lead to efficient allocation. The capital stock in the bubbly steady state is still lower than that in the efficient allocation.

How does the pledgeability parameter ξ affect the size of bubbles. Proposition 5 shows that a smaller ξ leads to a larger size of bubbles relative to capital. This is intuitive. If firms can only pledge a smaller amount of assets, they will face a tighter collateral constraint so that they have higher incentives to generate larger bubbles to finance investment.

5.2 Dynamics

Now, we study the stability of the two steady states and the local dynamics around these steady states. Since the equilibrium system (19)-(21) is three dimensional, we cannot use the phase diagram to analyze its stability. We thus consider a linearized system and obtain the following:

Proposition 6 *Suppose condition (35) holds. Then both the bubbly steady state* (B, Q_b, K_b) *and the bubbleless steady state* $(0, Q^*, K^*)$ *are local saddle points for the nonlinear system (19)-(21).*

More formally, in Appendix A, we prove that for the nonlinear system (19)-(21), there is a neighborhood $\mathcal{N} \subset \mathbb{R}^3_+$ of the bubbly steady state (B, Q_b, K_b) and a continuously differentiable function $\phi : \mathcal{N} \to \mathbb{R}^2$ such that given any K_0 there exists a unique solution (B_0, Q_0) to the equation $\phi(B_0, Q_0, K_0) = 0$ with $(B_0, Q_0, K_0) \in \mathcal{N}$, and (B_t, Q_t, K_t) converges to (B, Q_b, K_b) starting at (B_0, Q_0, K_0) as t approaches infinity. The set of points (B, Q, K) satisfying the equation $\phi(B, Q, K) = 0$ is a one dimensional stable manifold of the system. If the initial value (B_0, Q_0, K_0) is on the stable manifold, then the solution to the nonlinear system (19)-(21) is also on the stable manifold and converges to (B, Q_b, K_b) as t approaches infinity.

Although the bubbleless steady state $(0, Q^*, K^*)$ is also a local saddle point, the local dynamics around this steady state are different. In Appendix A, we prove that the stable manifold for the bubbleless steady state is two dimensional. Formally, there is a neighborhood $\mathcal{N}^* \subset \mathbb{R}^3_+$ of $(0, Q^*, K^*)$ and a continuously differentiable function $\phi^* : \mathcal{N}^* \to \mathbb{R}$ such that given any (B_0, K_0) there exists a unique solution Q_0 to the equation $\phi^*(B_0, Q_0, K_0) = 0$ with $(B_0, Q_0, K_0) \in \mathcal{N}$, and (B_t, Q_t, K_t) converges to $(0, Q^*, K^*)$ starting at (B_0, Q_0, K_0) as t approaches infinity. Intuitively, along the two-dimensional stable manifold, the bubbly equilibrium is asymptotically bubbleless in that bubbles will burst eventually.

6 Stochastic Bubbles

So far, we have focused on deterministic bubbles. Following Blanchard and Watson (1982) and Weil (1987), we now study stochastic bubbles. Consider the discrete-time economy described in Section 2. Suppose a bubble exists initially, $B_0 > 0$. In each time interval between t and $t + dt$, there is a constant probability θdt that the bubble will burst, $B_{t+dt} = 0$. Once it bursts, it will never be valued again so that $B_\tau = 0$ for all $\tau \geq t + dt$.[15] With the remaining probability $1 - \theta dt$, the bubble persists so that $B_{t+dt} > 0$. Later, we will take the continuous time limits as $dt \to 0$.

First, we consider the case in which the bubble has collapsed. This corresponds to the bubbleless equilibrium studied in Section 4. We use a variable with an asterisk (except for K_t) to denote its value in the bubbleless equilibrium. In particular, $V_t^*(K_t^j)$ denotes firm j's value function. In the continuous-time limit, (Q_t^*, K_t) satisfies the equilibrium system (20) and (21) with $B_t = 0$. We may express the solution for Q_t^* in a feedback form in that $Q_t^* = g(K_t)$ for some function g.

Next, we consider the case in which the bubble has not bursted. We assume that the debt contract is not contingent on extraneous beliefs or sunspots described earlier. Firms borrow and repay debt only when an investment opportunity arrives. The threat value to the lender or the stock market value of the collateralized assets is equal to the ex ante value before the realization of a sunspot, $e^{-rdt}V_{t+dt}(\xi K_t^j)(1 - \theta dt) + e^{-rdt}V_{t+dt}^*(\xi K_t^j)\theta dt$. Thus, the credit constraint is given by:

$$L_t^j \leq e^{-rdt}V_{t+dt}(\xi K_t^j)(1 - \theta dt) + e^{-rdt}V_{t+dt}^*(\xi K_t^j)\theta dt. \tag{36}$$

We write firm j's dynamic programming problem as follows:

$$\begin{aligned} V_t(K_t^j) \quad = \quad & \max \; R_t K_t^j dt - \pi I_t^j dt \\ & + e^{-rdt}(1 - \theta dt) V_{t+dt}((1 - \delta dt)K_t^j + I_t^j)\pi dt \\ & + e^{-rdt}(1 - \theta dt) V_{t+dt}((1 - \delta dt)K_t^j)(1 - \pi dt) \\ & + e^{-rdt}\theta dt \; V_{t+dt}^*((1 - \delta dt)K_t^j + I_t^j)\pi dt \\ & + e^{-rdt}\theta dt \; V_{t+dt}^*((1 - \delta dt)K_t^j)(1 - \pi dt) \end{aligned} \tag{37}$$

subject to (5) and (36).

[15] If a bubble reemerged in the future, it would have value today by the no-arbitrage asset-pricing equation. To generate recurrent bubbles and crashes, Miao and Wang (2011b) introduce firm entry and exit in the model. See Martin and Ventura (2010a) and Wang and Wen (2011) for other approaches.

We conjecture that the value function takes the form:

$$V_t(K_t^j) = v_t K_t^j + b_t, \tag{38}$$

where v_t and b_t are to be determined and are independent of K_t^j. As we have shown in Section 4, when the bubble bursts, the value function satisfies:

$$V_t^*(K_t^j) = v_t^* K_t^j. \tag{39}$$

After substituting the above two equations into (37) and simplifying, the firm's dynamic programming problem becomes:

$$v_t K_t^j + b_t = \max \; R_t K_t^j dt - \pi I_t^j dt + Q_t(1 - \delta dt) K_t^j + Q_t \pi I_t^j dt + B_t, \tag{40}$$

subject to

$$0 \leq I_t^j \leq R_t K_t^j + Q_t \xi K_t^j + B_t, \tag{41}$$

where we define $Q_t^* = e^{-rdt} v_{t+dt}^*$,

$$Q_t = e^{-rdt} \left[(1 - \theta dt) v_{t+dt} + \theta v_{t+dt}^* dt \right], \tag{42}$$

$$B_t = e^{-rdt} (1 - \theta dt) b_{t+dt}. \tag{43}$$

Suppose $Q_t > 1$. Then the optimal investment level achieves the upper bound in (41). Substituting this investment level into equation (40) and matching coefficients on the two sides of this equation, we obtain:

$$v_t = R_t dt + Q_t(1 - \delta dt) + \pi(Q_t - 1)(R_t + Q_t \xi) dt, \tag{44}$$

$$b_t = B_t + \pi(Q_t - 1) B_t dt. \tag{45}$$

As in Section 3, we conduct aggregation to obtain the discrete-time equilibrium system. We then take the continuous-time limits as $dt \to 0$ to obtain the following:

Proposition 7 *Suppose $Q_t > 1$. Before the bubble bursts, the equilibrium with stochastic bubbles (B_t, Q_t, K_t) satisfies the following system of differential equations:*

$$\dot{B}_t = (r + \theta) B_t - \pi(Q_t - 1) B_t, \tag{46}$$

$$\dot{Q}_t = (r + \delta + \theta) Q_t - \theta Q_t^* - R_t - \pi(Q_t - 1)(R_t + \xi Q_t), \tag{47}$$

and (21), where $R_t = \alpha K_t^{\alpha - 1}$ and $Q_t^* = g(K_t)$ is the capital price after the bubble bursts.

Equation (46) reveals that the rate of return on bubbles is equal to $r+\theta$, which is higher than the interest rate. This reflects risk premium because the stochastic bubble is risky. In general, it is hard to characterize the equilibrium with stochastic bubbles. In order to transparently illustrate the adverse impact of bubble bursting on the economy, we shall consider a simple type of equilibrium. Following Weil (1987) and Kocherlakota (2009), we study a stationary equilibrium with stochastic bubbles that has the following properties: The capital stock is constant at the value K_s over time before the bubble collapses. It continuously moves to the bubbleless steady state value K^* after the bubble collapses. The bubble is also constant at the value $B_s > 0$ before it collapses. It jumps to zero and then stays at this value after collapsing. The capital price is constant at the value Q_s before the bubble collapses. It jumps to the value $g(K)$ after the bubble collapses and then converges to the bubbleless steady-state value Q^* given in equation (27).

Our objective is to show the existence of (B_s, Q_s, K_s). By (46), we can show that

$$Q_s = \frac{r+\theta}{\pi} + 1. \tag{48}$$

Since $Q_s > 1$, we can apply Proposition 7 in some neighborhood of Q_s. Equation (47) implies that

$$0 = (r+\delta+\theta)Q_s - \theta g(K) - R - \pi(Q_s - 1)(R + \xi Q_s), \tag{49}$$

where $R = \alpha K^{\alpha-1}$. The solution to this equation gives K_s. Once we obtain K_s and Q_s, we use equation (31) to determine B_s.

The difficult part is to solve for K_s. In doing so, we define θ^* such that

$$\frac{r+\theta^*}{\pi} + 1 = \frac{\delta(1-\pi)}{\pi}\frac{1}{r+\xi} = Q^*. \tag{50}$$

That is, θ^* is the bursting probability such that the capital price in the stationary equilibrium with stochastic bubbles is the same as that in the bubbleless equilibrium.

Proposition 8 *Let condition (35) hold. If $0 < \theta < \theta^*$, then there exists a stationary equilibrium (B_s, Q_s, K_s) with stochastic bubbles such that $K_s > K^*$. In addition, if θ is sufficiently small, then consumption falls eventually after the bubble bursts.*

As in Weil (1987), a stationary equilibrium with stochastic bubbles exists if the probability that the bubble will burst is sufficiently small. In Weil's (1987) overlapping generations model, the capital stock and output eventually rise after the bubble collapses. In contrast to his result,

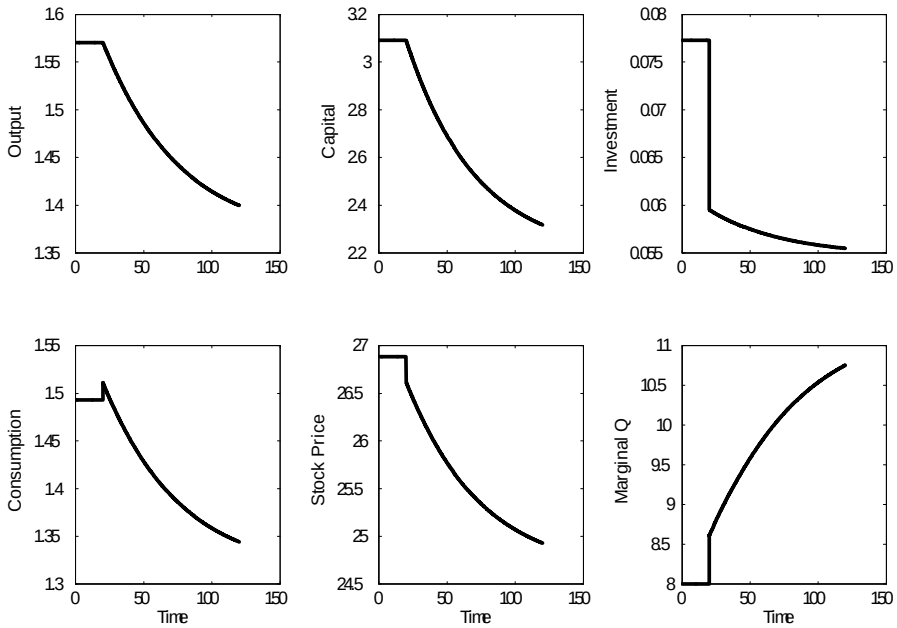

Figure 4: This figure plots the dynamics of the stationary equilibrium with stochastic bubbles. Assume that the bubble bursts at time $t = 20$. The parameter values are set as follows: $r = 0.02$, $\alpha = 0.4$, $\delta = 0.025$, $\theta = 0.05$, $\pi = 0.01$, and $\xi = 0.2$.

in our model the economy enters a recession after the bubble bursts in that consumption, capital and output all fall eventually. The intuition is that the collapse of the bubble tightens the collateral constraint and impairs investment efficiency.

Proposition 8 compares the economy before the bubble collapses with the economy after the bubble collapses only in the steady state. It would be interesting to see what happens along the transition path. Since analytical results are not available, we solve the transition path numerically and present the results in Figure 4.[16] In this numerical example, we assume that the bubble collapses at time $t = 20$. Immediately after the bubble collapses, investment falls discontinuously and then gradually decreases to its bubbleless steady-state level. But output and capital decrease continuously to their bubbleless steady-state levels since capital is predetermined and labor is exogenous. Consumption rises initially because of the fall of investment. But it quickly falls and then decreases to its bubbleless steady-state level. Importantly, the stock market crashes immediately after the bubble collapses in that the stock price drops discontinuously. Note that Tobin's marginal Q rises immediately after the bubble collapses and

[16] The parameter values for Figure 2 are not calibrated match the data since the model is stylized. Miao and Wang (2011b) develop a quantitative DSGE model to study how asset bubbles can explain US business cycles.

then gradually rises to its bubbleless steady-state value.[17] This reflects the fact that the capital stock gradually falls after the bubble collapses, causing the marginal product of capital to rise. To generate the fall of consumption and output on impact, we introduce endogenous capacity utilization in Appendix B. Following the collapse of bubbles, the capacity utilization rate falls because the value of installed capital rises. As a result, both output and consumption fall on impact. The collapse of bubbles generates a much severe recession.

7 Public Assets and Credit Policy

We have shown that the collapse of bubbles generates a recession. Is there a government policy that restores economic efficiency? The inefficiency of our model comes from the credit constraints. In our model, firms can use internal funds and external loans to finance investment. External loans are subject to collateral constraints. Bubbles help relax these constraints, while the collapse of bubbles tightens them.

Now suppose that the government can supply liquidity to the firms by issuing public bonds. These bonds are backed by lump-sum taxes.[18] We consider unbacked assets in Section 7.3. We first suppose that households can buy and sell these bonds without any trading frictions. We then relax this assumption in Section 7.3. Firms can use public bonds as collateral to relax their collateral constraints. They can also buy and sell these assets to finance investment. Let the quantity of government bonds supplied to the firms be M_t and the bond price be P_t. We start with the discrete-time environment described in Section 2. The value of the government assets satisfies:

$$M_t P_t = T_t dt + M_{t+dt} P_t, \tag{51}$$

where T_t denotes lump-sum taxes. Taking the continuous-time limits yields:

$$\dot{M}_t P_t = -T_t. \tag{52}$$

It is more convenient to define $D_t = P_t M_t$. Then use the fact that $\dot{D}_t = \dot{P}_t M_t + \dot{M}_t P_t$ to rewrite (52) as:

$$\dot{D}_t - \dot{P}_t M_t = \dot{D}_t - \frac{\dot{P}_t}{P_t} D_t = -T_t \text{ if } P_t > 0. \tag{53}$$

[17] Note that marginal Q is not equal to average Q given the constant-returns-to-scale assumption in our model because of the presence of stock-price bubbles.

[18] As an idealized benchmark, we ignore the issues of moral hazard and distortional taxes.

Since households are assumed to be risk neutral, we immediately obtain the asset pricing equation for the government bonds: $P_t = e^{-rdt} P_{t+dt}$. Taking the continuous-time limit yields:

$$\dot{P}_t = rP_t, \tag{54}$$

which implies that the growth rate of the government bond price is equal to the interest rate. Now we turn to firms' optimization problem below.

7.1 Equilibrium after the Bubble Bursts

We solve firms' dynamic optimization problem by dynamic programming. We start with the case in which the stock market bubble has collapsed. Because firms can trade public assets, holdings of government bonds are another state variable. We write firm j's dynamic programming problem as follows:

$$\begin{aligned} V_t^*(K_t^j, M_t^j) \quad = \quad & \max_{I_t^j, M_{t+dt}^j} \quad R_t K_t^j dt - \pi I_t^j dt + P_t(M_t^j - M_{t+dt}^j) \\ & + e^{-rdt} V_{t+dt}^*((1-\delta dt)\, K_t^j + I_t^j, M_{t+dt}^j)\ \pi dt \\ & + e^{-rdt} V_{t+dt}^*((1-\delta dt)\, K_t^j, M_{t+dt}^j)\ (1-\pi dt), \end{aligned} \tag{55}$$

subject to (5) and

$$L_t^j \leq e^{-rdt} V_{t+dt}^* \left(\xi K_t^j, 0\right) + P_t M_t^j, \tag{56}$$

$$M_{t+dt}^j \geq 0, \tag{57}$$

where M_t^j denotes the amount of government assets held by firm j. In equilibrium, $\int M_t^j dj = M_t$. Equation (56) indicates that firms use government assets as collateral. The expression $e^{-rdt} V_{t+dt}^* \left(\xi K_t^j, 0\right)$ gives the market value of the collateralized assets ξK_t^j. Equation (57) is a short sales constraint, which rules out Ponzi schemes for public assets. It turns out that this constraint will never bind for firms.

As in Section 3, we conjecture the value function takes the form:

$$V_t^*(K_t^j, M_t^j) = v_t^* K_t^j + v_t^{*M} M_t^j, \tag{58}$$

where v_t^* and v_t^{*M} are to be determined variables, that are independent of K_t^j or M_t^j. Because bubbles have collapsed, there is no bubble term in this conjecture. We define

$$Q_t^* = e^{-rdt} v_{t+dt}^*, \quad Q_t^{*M} = e^{-rdt} v_{t+dt}^{*M}.$$

We can then rewrite (55) as:

$$\begin{aligned} v_t^* K_t^j + v_t^{*M} M_t^j &= \max\ R_t K_t^j dt - \pi I_t^j dt + P_t(M_t^j - M_{t+dt}^j) \\ &\quad + Q_t^*(1-\delta dt)K_t^j + \pi Q_t^* I_t^j dt + Q_t^{*M} M_{t+dt}^j, \end{aligned} \tag{59}$$

subject to $M_{t+dt}^j \geq 0$ and

$$0 \leq I_t^j \leq R_t K_t^j + \xi Q_t^* K_t^j + P_t M_t^j, \tag{60}$$

When $Q_t^* > 1$, optimal investment achieves the upper bound. For an interior solution for the optimal holdings of government assets to exist, we must have $P_t = Q_t^{*M}$. Matching coefficients of K_t^j and M_t^j as well as the constant terms on the two sides of (59), we obtain (14), (15), and

$$v_t^{*M} = P_t + (Q_t^* - 1)P_t \pi dt.$$

As in Proposition 2, we may conduct aggregation and derive the equilibrium system for (P_t, Q_t^*, K_t) in the discrete time case. As in Proposition 3, the continuous-time limits satisfy the following differential equations:

$$\dot{P}_t = rP_t - P_t \pi (Q_t^* - 1), \tag{61}$$

$$\dot{K}_t = -\delta K_t + \pi(R_t K_t + \xi Q_t^* K_t + P_t M_t),\ K_0 \text{ given}, \tag{62}$$

and an equation analogous to (20) for Q_t^*. In addition, the transversality condition

$$\lim_{T\to\infty} e^{-rT} P_T M_T = 0,$$

and other transversality conditions for K_t and B_t as described in Proposition 3 must be satisfied. Here, we omit the detailed derivation of these conditions and the above differential equations.

Equation (61) is an asset pricing equation for the firms' trading. It is identical to the asset pricing equation (19) for the bubble. This is because the government bonds and the bubble on firm assets play the same role for the firms in that both of them can be used to relax the credit constraints. The dividend yield of the government bonds to the firms is equal to $\pi(Q_t^* - 1)$ when the bond price is positive. By contrast, there is no dividend yield to the households, as revealed by equation (54).

Comparing (54) with (61), we deduce that $Q_t^* = 1$. Substituting it into equation (20) reveals that $R_t = r + \delta$. This equation gives the efficient capital stock K_E for all time t. To support this capital stock in equilibrium, the value of the government debt $D_t = P_t M_t$ must satisfy equation (62) for $K_t = K_E$. Solving yields:

$$D_t = D \equiv K_E \left(\delta \frac{1-\pi}{\pi} - r - \xi \right) > 0 \tag{63}$$

By equations (54) and (53), we deduce that the lump-sum taxes must satisfy $T_t = T \equiv rD$ for all t.

7.2 Equilibrium before the Bubble Bursts

Now, we turn to the equilibrium before the bubble bursts. We have to modify the dynamic programming problem (37) by incorporating the trading of government bonds. By an analysis similar to that in the previous subsection and in Section 6, we can derive the continuous-time equilibrium system for (P_t, B_t, Q_t, K_t) before the bursting of the stock-price bubble. This system is given by equations (46), (47) and

$$\dot{P}_t = (r + \theta) P_t - P_t \pi (Q_t - 1), \tag{64}$$

$$\dot{K}_t = -\delta K_t + \pi (R_t K_t + \xi Q_t K_t + B_t + P_t M_t), \ K_0 \text{ given.} \tag{65}$$

By a no-arbitrage argument similar to that in the previous subsection, we deduce that $Q_t = 1$. By equation (47) and $Q_t^* = 1$, we deduce that $R_t = r + \delta$, which gives the efficient capital stock K_E. In addition, $Q_t = 1$ and equation (46) imply that $B_t = 0$ for all t. The bubble on the firm assets cannot be sustained in equilibrium because its dividend yield is zero and thus its growth rate is equal to $r + \theta$, which is higher than the zero rate of economic growth. Equation (65) gives the value of the government debt $D_t = P_t M_t$ that supports the above efficient allocation.

We summarize the above analysis in the following proposition and relegate its proof to Appendix A.

Proposition 9 *Suppose assumption (35) holds. Let the government issue a constant value D of government debt given by (63), which is backed by lump-sum taxes $T_t = T \equiv rD$ for all t. Then this credit policy will eliminate the bubble on firm assets and make the economy achieve the efficient allocation.*[19]

This proposition indicates that the government can design a credit policy that eliminates bubbles and achieves the efficient allocation. The key intuition is that the government may provide sufficient liquidity to firms so that firms do not need to rely on bubbles to relax credit constraints. The government plays the role of financial intermediaries by transferring funds from

[19] Note that for the proof of this proposition, we have used the household pricing equation (54) when households do not face trading frictions. Even though they face short sales constraints, we can use (53) to derive this equation, given constant D and $T_t = rD$. Thus, this proposition holds true given this assumption.

households to firms directly so that firms can overcome credit constraints. The government bond is a store of value and can also generate dividends to firms. The dividend yield is equal to the net benefit from new investment. For households, the government bond is just a store of value. No arbitrage forces the dividend yield to zero, which implies that the capital price must be equal to one. As a result, the economy can achieve the efficient allocation.

To implement the above policy. The government constantly retires the public bonds at the interest rate in order to keep the total bond value constant. To back the government bonds, the government levies constant lump-sum taxes equal to the interest payments of bonds.

7.3 Bubbles on Intrinsically Useless Assets

An important assumption of the above credit policy is that the public bonds must be backed by lump-sum taxes. What will happen if public bonds are unbacked assets? In this case, equation (51) implies that M_t is constant over time since $T_t = 0$. We thus normalize $M_t = 1$ for all t. Our previous asset pricing equations (54) (61), and (64) for public bonds still apply here if households do not face trading frictions. Thus, if $P_t > 0$, then $Q_t = Q_t^* = 1$, which implies that $K_t = K_E$ for all t. In this case, the capital accumulation equations (62) and (65) imply that the public bond price P_t must be constant over time, contradicting with the asset pricing equations for bonds. Thus, in equilibrium $P_t = 0$. The intuition is that the public bond is a bubble when it is an unbacked asset. Its rate of return or its growth rate is equal to the riskfree interest rate which is higher than the zero economic growth rate. Thus, the bubble cannot sustain in equilibrium.

A bubble on unbacked public assets (or any intrinsically useless assets) can exist when households face short sales constraints on these assets (e.g., Kocherlakota (1992, 1999) and Wang and Wen (2011)). In this case, the asset-pricing equation (54) becomes:

$$\mu_t + \dot{P}_t = rP_t, \tag{66}$$

where μ_t is the Lagrange multiplier associated with the short sales constraint. Equations (61) and (64) still hold. The bubble solution $P_t > 0$ is possible if $\mu_t > 0$. Firms prefer to demand public bonds and use these bonds to finance investment. Households prefer to sell public bonds as much as possible until the short sales constraints bind. These intrinsically useless assets have value because they can be used to relax credit constraints and improve investment efficiency.

In our model, the policymaker may create a bubble to boost the economy by issuing unbacked public assets when it enters a recession after the collapse of stock-price bubbles. This

policy is dangerous to the economy. To see this, we derive the steady state with the bubble on these public assets. By equation (61), the steady-state Q^* is identical with the value given in (33). Using the steady-state equation for Q^* in (23), we can show that the steady-state capital stock is given by (34). We then use equation (62) to derive the steady-state ratio of the total value of the public assets PM to the capital stock, which is identical to the value given in (32). Given the steady-state supply of public assets M, we can determine the steady-state price of public assets P. Note that the steady-state allocation for this economy is identical to that for the economy with stock-price bubbles studied in Section 5.1. The allocation is still inefficient. In addition, the bubble on public assets can burst as long as everyone believes that the probability of bursting is high enough as shown in Section 6. After the burst of the bubble on unbacked public assets, the economy enters a recession again.

We stress that it is possible to have bubbles on both stock prices and intrinsically useless assets. As equation (65) shows, equilibrium only determines the aggregate bubble $B_t + P_t M_t$. But the decomposition is indeterminate because both B_t and P_t satisfy the same equilibrium conditions and follow the same dynamics (see (46) and (64)). The equilibrium real allocation is independent of the decomposition. This result is analogous to that discussed in Section 5 of Tirole (1985).

8 Conclusion

In this paper, we provide an infinite-horizon model of a production economy with bubbles, in which firms meet stochastic investment opportunities and face credit constraints. Capital is not only an input for production, but also serves as collateral. We show that bubbles on this reproducible asset may arise, which relax collateral constraints and improve investment efficiency. The collapse of bubbles leads to a recession, even though there is no exogenous shock to the fundamentals of the economy. Immediately after the collapse, investment falls discontinuously and the stock market crashes in that the stock price falls discontinuously. In the long run, output, investment, consumption, and capital all fall to their bubbleless steady-state values. We show that there is a credit policy that can eliminate the bubble on firm assets and can achieve the efficient allocation.

We focus on firms' credit constraints and consider a deterministic economy in which all firm assets are publicly traded in a stock market. We study bubbles on these assets. Thus, our analysis provides a theory of the creation and collapse of stock price bubbles driven by

the credit market conditions. Our analysis differs from most studies in the existing literature that analyze bubbles on intrinsically useless assets or on assets with exogenously given rents or dividends. Our model can incorporate this type of bubbles and thus provides a unified framework to study asset bubbles with firm heterogeneity and borrowing constraints. In future research, it would be interesting to consider households' endogenous borrowing constraints or incomplete markets economies and then study the role of bubbles in this kind of environments. It would also be interesting to study how bubbles contribute to business cycles in a quantitative dynamic stochastic general equilibrium model (see Miao and Wang (2011b)). Finally, there is no economic growth in the present paper. Miao and Wang (2011a) extend the present paper to study endogenous growth.[20]

[20] See Grossman and Yanagawa (1993), Olivier (2000), Caballero, Farhi and Hammour (2006), Hirano and Yanagawa (2010) and Martin and Venture (2009) for models of asset bubbles and economic growth.

Appendices

A Proofs

Proof of Proposition 1: Substituting the conjecture (10) into (4) and (6) yields:

$$\begin{aligned} v_t K_t^j + b_t &= \max\ R_t K_t^j dt - \pi I_t^j dt + e^{-rdt} v_{t+dt} \left((1-\delta dt) K_t^j + I_t^j \right) \pi dt \\ &\quad + e^{-rdt} v_{t+dt} (1-\delta dt) K_t^j (1-\pi dt) + B_t, \end{aligned} \tag{A.1}$$

$$L_t^j \leq \xi e^{-rdt} v_{t+dt} K_t^j + B_t, \tag{A.2}$$

where B_t is defined in (12). We combine (5) and (A.2) to obtain:

$$0 \leq I_t^j \leq R_t K_t^j + \xi e^{-rdt} v_{t+dt} K_t^j + B_t. \tag{A.3}$$

Let Q_t be the Lagrange multiplier associated with (3) for the case with the arrival of the investment opportunity. The first-order condition with respect to K_{t+dt}^j delivers equation (13). When $Q_t > 1$, we obtain the optimal investment rule in (11). Plugging (11) and (3) into the Bellman equation (A.1) and matching coefficients of K_t^j and the terms unrelated to K_t^j, we obtain (14) and (15). Q.E.D.

Proof of Proposition 2: Using the optimal investment rule in (11) and aggregating equation (3), we obtain the aggregate capital accumulation equation (18) and the aggregate investment equation (22). Substituting (15) into (12) yields (16). Substituting (14) into (13) yields (17). The first-order condition for the static labor choice problem (1) gives $w_t = (1-\alpha)(K_t^j/N_t^j)^\alpha$. We then obtain (2) and $K_t^j = N_t^j (w_t/(1-\alpha))^{1/\alpha}$. Thus, the capital-labor ratio is identical for each firm. Aggregating yields $K_t = N_t (w_t/(1-\alpha))^{1/\alpha}$. Using this equation to substitute out w_t in (2) yields $R_t = \alpha K_t^{\alpha-1} N_t^{1-\alpha} = \alpha K_t^{\alpha-1}$ since $N_t = 1$ in equilibrium. Aggregate output satisfies

$$Y_t = \int (K_t^j)^\alpha (N_t^j)^{1-\alpha} dj = \int (K_t^j/N_t^j)^\alpha N_t^j dj = (K_t^j/N_t^j)^\alpha \int N_t^j dj = K_t^\alpha N_t^{1-\alpha}.$$

This completes the proof. Q.E.D.

Proof of Proposition 3: By equation (18),

$$\frac{K_{t+dt} - K_t}{dt} = -\delta K_t + [R_t + \xi Q_t K_t + B_t]\ \pi.$$

Taking limit as $dt \to 0$ yields equation (21). Using the approximation $e^{rdt} = 1 + rdt$ in equation (16) yields:

$$B_t(1 + rdt) = B_{t+dt}\left[1 + \pi(Q_{t+dt} - 1)dt\right].$$

Simplifying yields:

$$\frac{B_t - B_{t+dt}}{dt} + rB_t = B_{t+dt}\pi(Q_{t+dt} - 1).$$

Taking limits as $dt \to 0$ yields equation (19). Finally, we approximate equation (17) by:

$$Q_t(1 + rdt) = R_{t+dt}dt + (1 - \delta dt)Q_{t+dt} + \left(R_{t+dt} + \xi Q_{t+dt}\right)\left(Q_{t+dt} - 1\right)\pi dt.$$

Simplifying yields:

$$\frac{Q_t - Q_{t+dt}}{dt} + rQ_t = R_{t+dt} - \delta Q_{t+dt} + \left(R_{t+dt} + \xi Q_{t+dt}\right)\left(Q_{t+dt} - 1\right)\pi.$$

Taking limit as $dt \to 0$ yields equation (20).

We may start with a continuous-time formulation directly. The Bellman equation in continuous time satisfies:

$$\begin{aligned} rV\left(K^j, S\right) &= \max_{I^j} RK^j - \pi I^j + \pi\left[V\left(K^j + I^j, S\right) - V\left(K^j, S\right)\right] \\ &\quad -\delta K^j V_K\left(K^j, S\right) + V_S\left(K^j, S\right)\dot{S}, \end{aligned}$$

where $S = (B, Q)$ represents the vector of aggregate state variables. We may derive this Bellman equation by taking limits in (4) as $dt \to 0$. Conjecture $V\left(K^j, B, Q\right) = QK^j + B$. We can then solve the above Bellman equation. After aggregation, we can derive the system of differential equations in the proposition. Q.E.D.

Proof of Proposition 4: (i) The social planner solves the following problem:

$$\max_{I_t} \int_0^\infty e^{-rt}\left(K_t^\alpha - \pi I_t\right)dt$$

subject to

$$\dot{K}_t = -\delta K_t + \pi I_t,\ K_0 \text{ given}$$

where K_t is the aggregate capital stock and I_t is the investment level for each firm with the arrival of the investment opportunity. From this problem, we can derive the efficient capital stock K_E, which satisfies $\alpha\left(K_E\right)^{\alpha-1} = r + \delta$. The efficient output, investment and consumption levels are given by $Y_E = \left(K_E\right)^\alpha$, $I_E = \delta/\pi K_E$, and $C_E = \left(K_E\right)^\alpha - \delta K_E$, respectively.

From the proof of Proposition 1, we can rewrite (A.1) as:

$$v_t K_t^j = \max \; R_t K_t^j dt - \pi I_t^j dt + Q_t(1-\delta dt)K_t^j + Q_t \pi I_t^j dt. \tag{A.4}$$

Suppose assumption (25) holds. We conjecture $Q^* = 1$ and $Q_t = 1$. Substituting this conjecture into the above equation and matching coefficients of K_t^j give:

$$v_t = R_t dt + 1 - \delta dt.$$

Since $Q_t = e^{-rdt} v_{t+dt} = 1$, we have $e^{rdt} = R_{t+dt} dt + 1 - \delta dt$. Approximating this equation yields:

$$1 + rdt = R_{t+dt} dt + 1 - \delta dt.$$

Taking limits as $dt \to 0$ gives $R_t = r + \delta = \alpha K_t^{*\alpha-1}$. Thus, $K_t^* = K_E$. Given this constant capital stock for all firms, the optimal investment level satisfies $\delta K_t^* = \pi I_t^*$. Thus, $I_t^*/K_t^* = \delta/\pi$. We can easily check that assumption (25) implies that

$$\frac{\delta}{\pi} = I_t^*/K_t^* \leq R_t + \xi = r + \delta + \xi.$$

Thus, the investment constraint (5) or (A.3) is satisfied for $Q_t = 1$ and $B_t = 0$. We conclude that the solutions $Q_t = 1$, $K_t^* = K_E$, and $I_t^*/K_t^* = \delta/\pi$ give the bubbleless equilibrium, which also delivers the efficient allocation.

(ii) Suppose (26) holds. Conjecture $Q_t > 1$ in some neighborhood of the bubbleless steady state. We can then apply Proposition 3 and derive the steady-state equations (23) and (24). From these equation, we obtain the steady-state solution Q^* and K^* in (27) and (28), respectively. Assumption (26) implies that $Q^* > 1$. By continuity, $Q_t > 1$ in some neighborhood of (Q^*, K^*). This verifies our conjecture. Q.E.D.

Proof of Proposition 5: Solving equations (23), (30), (31) yields equations (32)-(34). By (32), $B > 0$ if and only if (35) holds. From (27) and (33), we deduce that $Q_b < Q^*$. Using condition (35), it is straightforward to check that $K_{GR} > K_E > K_b > K^*$. From (32), it is also straightforward to verify that the bubble-asset ratio B/K_b decreases with ξ. Q.E.D.

Proof of Proposition 6: First, we consider the log-linearized system around the bubbly steady state (B, Q_b, K_b). We use $\hat{X}_t$ to denote the percentage deviation from the steady state value for any variable X_t, i.e., $\hat{X}_t = \ln X_t - \ln X$. We can show that the log-linearized system

is given by:

$$\begin{bmatrix} d\hat{B}_t/dt \\ d\hat{Q}_t/dt \\ d\hat{K}_t/dt \end{bmatrix} = A \begin{bmatrix} \hat{B}_t \\ \hat{Q}_t \\ \hat{K}_t \end{bmatrix},$$

where

$$A = \begin{bmatrix} 0 & -(r+\pi) & 0 \\ 0 & \delta - \frac{(r+\delta+\xi)(r+\pi)}{1+r} & [(1-\xi)r+\delta](1-\alpha) \\ \pi B/K_b & \xi(r+\pi) & -(\pi R_b(1-\alpha)+\pi B/K_b) \end{bmatrix}. \tag{A.5}$$

We denote this matrix by:

$$A = \begin{bmatrix} 0 & a & 0 \\ 0 & b & c \\ d & e & f \end{bmatrix},$$

where we deduce from (A.5) that $a < 0$, $b > 0$, $c > 0$, $d > 0$, $e > 0$, and $f < 0$. We compute the characteristic equation for the matrix A:

$$F(x) \equiv x^3 - (b+f)x^2 + (bf - ce)x - acd = 0. \tag{A.6}$$

We observe that $F(0) = -acd > 0$ and $F(-\infty) = -\infty$. Thus, there exists a negative root to the above equation, denoted by $\lambda_1 < 0$. Let the other two roots be λ_2 and λ_3. We rewrite $F(x)$ as:

$$\begin{aligned} F(x) &= (x-\lambda_1)(x-\lambda_2)(x-\lambda_3) \\ &= x^3 - (\lambda_1+\lambda_2+\lambda_3)x^2 + (\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_2\lambda_3)x - \lambda_1\lambda_2\lambda_3. \end{aligned} \tag{A.7}$$

Matching terms in equations (A.6) and (A.7) yields $\lambda_1\lambda_2\lambda_3 = acd < 0$ and

$$\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 = bf - cd < 0. \tag{A.8}$$

We consider two cases. (i) If λ_2 and λ_3 are two real roots, then it follows from $\lambda_1 < 0$ that λ_2 and λ_3 must have the same sign. Suppose $\lambda_2 < 0$ and $\lambda_3 < 0$, we then have $\lambda_1\lambda_2 > 0$ and $\lambda_1\lambda_3 > 0$. This implies that $\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 > 0$, which contradicts equation (A.8). Thus, we must have $\lambda_2 > 0$ and $\lambda_3 > 0$.

(ii) If either λ_2 or λ_3 is complex, then the other must also be complex. Let

$$\lambda_2 = g + hi \text{ and } \lambda_3 = g - hi,$$

where g and h are some real numbers. We can show that

$$\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 = 2g\lambda_1 + g^2 + h^2.$$

Since $\lambda_1 < 0$, the above equation and equation (A.8) imply that $g > 0$.

From the above analysis, we conclude that the matrix A has one negative eigenvalues and the other two eigenvalues are either positive real numbers or complex numbers with positive real part. As a result, the bubbly steady state is a local saddle point and the stable manifold is one dimensional.

Next, we consider the local dynamics around the bubbleless steady state $(0, Q^*, K^*)$. We linearize B_t around zero and log-linearize Q_t and K_t and obtain linearized system:

$$\begin{bmatrix} dB_t/dt \\ d\hat{Q}_t/dt \\ d\hat{K}_t/dt \end{bmatrix} = J \begin{bmatrix} B_t \\ \hat{Q}_t \\ \hat{K}_t \end{bmatrix},$$

where

$$J = \begin{bmatrix} r - \pi(Q^* - 1) & 0 & 0 \\ 0 & a & b \\ \frac{\pi}{K^*} & c & d \end{bmatrix},$$

where

$$\begin{aligned} a &= \frac{R^*}{Q^*}[1 + \pi(Q^* - 1)] - (\frac{R^*}{Q^*} + \xi)\pi Q^*, \\ b &= \frac{R^*}{Q^*}[1 + \pi(Q^* - 1)](1 - \alpha) > 0, \\ c &= \pi\xi Q^* > 0, \\ d &= \pi R^*[\alpha - 1] < 0. \end{aligned}$$

Using a similar method for the bubbly steady state, we analyze the three eigenvalues of the matrix J. One eigenvalue, denoted by λ_1, is equal to $r - \pi(Q^* - 1) < 0$ and the other two, denoted by λ_2 and λ_3, satisfy

$$\lambda_2\lambda_3 = ad - bc. \tag{A.9}$$

Notice that we have

$$\frac{a}{b} = \frac{1}{1-\alpha}\left[\frac{1-\pi}{1+\pi(Q^*-1)} - \xi\frac{Q^*}{R^*}\frac{\pi Q^*}{1+\pi(Q^*-1)}\right],$$

and

$$\frac{c}{d} = -\frac{\xi Q^*}{R^*}\frac{1}{1-\alpha}.$$

So we have

$$\frac{a}{b} - \frac{c}{d} > 0 \text{ or } \frac{a}{b} > \frac{c}{d}.$$

Since $b > 0$ and $d < 0$, we deduce that $ad < cb$. It follows from (A.9) that $\lambda_2\lambda_3 < 0$, implying that λ_2 and λ_3 must be two real numbers with opposite signs. We conclude that the bubbleless steady state is a local saddle point and the stable manifold is two dimensional. Q.E.D.

Proof of Proposition 7: As we discussed in the main text, we may derive equations (44) and (45). Substituting equation (44) into (42) and using the definition $Q_t^* = e^{-rdt} v_{t+dt}^*$, we can derive that:

$$\begin{aligned} Q_t &= \theta Q_t^* dt + e^{-rdt}(1-\theta dt)[R_{t+dt}dt + Q_{t+dt}(1-\delta dt) \\ &\quad + \pi(Q_{t+dt}-1)(R_{t+dt}+Q_{t+dt}\xi)dt]. \end{aligned} \tag{A.10}$$

Using the approximation $e^{-rdt} = 1 - rdt$ and removing all terms that have orders at least dt^2, we approximate the above equation by:

$$\begin{aligned} Q_t - Q_{t+dt} &= \theta Q_t^* dt + R_{t+dt}dt - \delta Q_{t+dt}dt + \pi(Q_{t+dt}-1)(R_{t+dt}+\xi Q_{t+dt})dt \\ &\quad - (r+\theta)\, Q_{t+dt}dt. \end{aligned} \tag{A.11}$$

Dividing by dt on the two sides and taking limits as $dt \to 0$, we obtain:

$$-\dot{Q}_t = R_t - (r+\delta+\theta)Q_t + \theta Q_t^* + \pi(Q_t - 1)(R_t + \xi Q_t), \tag{A.12}$$

which gives equation (47). Similarly, substituting equation (45) into (43) and taking limits, we can derive equation (46). Q.E.D.

Proof of Proposition 8: Let $Q(\theta)$ be the expression on the right-hand side of equation (48). We then use this equation to rewrite equation (49) as:

$$\alpha K^{\alpha-1}(1+r+\theta) - (r+\delta+\theta)Q(\theta) + \theta g(K) + (r+\theta)\xi Q(\theta) = 0.$$

Define the function $F(K;\theta)$ as the expression on the left-hand side of the above equation. Notice $Q(\theta^*) = Q^* = g(K^*)$ by definition and $Q(0) = Q_b$ where Q_b is given in (33). The condition (35) ensures the existence of the bubbly steady-state value Q_b and the bubbleless steady-state values Q^* and K^*.

Define

$$K_{\max} = \max_{0\le\theta\le\theta^*} \left[\frac{(r+\delta+\theta-(r+\theta)\xi)Q(\theta) - \theta Q^*}{\alpha(1+r+\theta)}\right]^{\frac{1}{\alpha-1}}.$$

By (34), we can show that

$$K_b = \left[\frac{(r+\delta-r\xi)Q(0)}{\alpha(1+r)}\right]^{\frac{1}{\alpha-1}}.$$

Thus, we have $K_{\max} \ge K_b$ and hence $K_{\max} > K^*$. We want to prove that

$$F(K^*;\theta) > 0,\ F(K_{\max};\theta) < 0,$$

for $\theta \in (0, \theta^*)$. If this true, then it follows from the intermediate value theorem that there exists a solution K_s to $F(K;\theta) = 0$ such that $K_s \in (K^*, K_{\max})$.

First, notice that

$$\begin{aligned} F(K^*, 0) &= \alpha K^{*\alpha-1}(1+r) - r(1-\xi)Q_b - \delta Q_b \\ &> \alpha K_b^{\alpha-1}(1+r) - r(1-\xi)Q_b - \delta Q_b \\ &= 0, \end{aligned}$$

and

$$F(K^*, \theta^*) = 0.$$

We can verify that $F(K;\theta)$ is concave in θ for any fixed K. Thus, for all $0 < \theta < \theta^*$,

$$\begin{aligned} F(K^*;\theta) &= F\left(K^*, (1-\frac{\theta}{\theta^*})0 + \frac{\theta}{\theta^*}\theta^*\right) \\ &> (1-\frac{\theta}{\theta^*})F(K^*, 0) + \frac{\theta}{\theta^*}F(K^*, \theta^*) \\ &> 0. \end{aligned}$$

Next, for $K \in (K^*, K_{\max})$, we derive the following:

$$\begin{aligned} F(K_{\max};\theta) &= \alpha K_{\max}^{\alpha-1}(1+r+\theta) - (r+\delta+\theta)Q(\theta) + \theta g(K_{\max}) + (r+\theta)\xi Q(\theta) \\ &< \alpha K_{\max}^{\alpha-1}(1+r+\theta) - (r+\delta+\theta)Q(\theta) + \theta g(K^*) + (r+\theta)\xi Q(\theta) \\ &< 0, \end{aligned}$$

where the first inequality follows from the fact that the saddle path for the bubbleless equilibrium is downward sloping as illustrated in Figure 3 so that $g(K_{\max}) < g(K^*)$, and the second inequality follows from the definition of $K_{\max}$ and the fact that $g(K^*) = Q^*$.

Finally, note that $Q(\theta) < Q^*$ for $0 < \theta < \theta^*$. We use equation (31) and $K_s > K^*$ to deduce that

$$\begin{aligned} \frac{B_s}{K_s} &= \frac{\delta}{\pi} - \alpha K_s^{\alpha-1} - \xi Q(\theta) \\ &> \frac{\delta}{\pi} - \alpha K^{*\alpha-1} - \xi Q^* \\ &= 0. \end{aligned}$$

This completes the proof of the existence of stationary equilibrium with stochastic bubbles (B_s, Q_s, K_s).

When $\theta = 0$, the bubble never bursts and hence $K_s = K_b$. When θ is sufficiently small, K_s is close to K_b by continuity. Since K_b is less than the golden rule capital stock K_{GR}, $K_s < K_{GR}$ when θ is sufficiently small. Since $K^\alpha - \delta K$ is increasing for all $K < K_{GR}$, we deduce that $K_s^\alpha - \delta K_s > K^{*\alpha} - \delta K^*$. This implies that the consumption level before the bubble collapses is higher than the consumption level in the steady state after the bubble collapses. Q.E.D.

Proof of Proposition 9: We write the firm's dynamic programming before the bubble collapses as:

$$\begin{aligned} V_t(K_t^j, M_t^j) \;=\; & \max\; R_t K_t^j dt - \pi I_t^j dt + P_t M_t^j - P_t M_{t+dt}^j \\ & + e^{-rdt}\,(1-\theta dt)\, V_{t+dt}((1-\delta dt)K_t^j + I_t^j, M_{t+dt}^j)\pi dt \\ & + e^{-rdt}\,(1-\theta dt)\, V_{t+dt}((1-\delta dt)K_t^j, M_{t+dt}^j)\,(1-\pi dt) \\ & + e^{-rdt}\theta dt\; V_{t+dt}^*((1-\delta dt)K_t^j + I_t^j, M_{t+dt}^j)\pi dt \\ & + e^{-rdt}\theta dt\; V_{t+dt}^*((1-\delta dt)K_t^j, M_{t+dt}^j)\,(1-\pi dt)\,, \end{aligned} \tag{A.13}$$

subject to (5), (57), and

$$L_t^j \le e^{-rdt} V_{t+dt}(\xi K_t^j, 0)\,(1-\theta dt) + e^{-rdt} V_{t+dt}^*(\xi K_t^j, 0)\theta dt + P_t M_t. \tag{A.14}$$

We conjecture that the value function takes the form:

$$V_t\left(K_t^j, M_t^j\right) = v_t K_t^j + v_t^M M_t^j + b_t,$$

where v_t, v_t^M, and b_t are to be determined variables independent of j. Define Q_t and B_t as in (42) and (43), respectively, and define

$$Q_t^M = e^{-rdt}\left[(1-\theta dt)\, v_{t+dt}^M + v_{t+dt}^{*M}\theta dt\right].$$

By an analysis similar to that in Section 7.1, we can derive the continuous-time limiting system for (P_t, B_t, Q_t, K_t) given in Section 7.2. Finally, we follow the procedure described there to establish Proposition 9. Q.E.D.

B Capacity Utilization and Stochastic Bubbles

In the main text, we have shown that consumption immediately rises after the burst of bubbles. This result seems counterfactual. In this appendix, we show that our model can generate a

more realistic prediction after introducing variable capacity utilization. Assume that each firm j's production is given by

$$Y_t^j = (u_t^j K_t^j)^\alpha \left(N_t^j\right)^{1-\alpha}, \tag{B.1}$$

where u_t^j represents the capacity utilization rate. Assume that the firm's deprecation rate between period t to period $t + dt$ is given by

$$\delta_t^j = \varphi(u_t^j), \tag{B.2}$$

where φ is an increasing and convex function. Suppose that the capacity utilization decision is made before the arrival of investment opportunities. This assumption will ensure that the capacity utilization rates are equal across all the firms, facilitating aggregation.

Solving the static labor choice yields the operating profits:

$$\max_{N_t^j} \ (u_t^j K_t^j)^\alpha \left(N_t^j\right)^{1-\alpha} - w_t N_t^j = u_t^j R_t K_t^j. \tag{B.3}$$

We can then repeat the analysis in the main text by replacing equation (1) with equation (B.3) and replacing δ with δ_t^j in (B.2). We summarize the solution to the bubbleless equilibrium in the following:

Proposition 10 *Suppose $Q_t > 1$. The bubbleless equilibrium with capacity utilization (Q_t, u_t, K_t) satisfies the following system of differential equations:*

$$\dot{Q}_t = (r + \delta_t)Q_t - u_t R_t - \pi(Q_t - 1)(u_t R_t + \xi Q_t), \tag{B.4}$$

$$\dot{K}_t = -\delta_t K_t + (u_t R_t K_t + \xi Q_t K_t)\pi, \tag{B.5}$$

where $\delta_t = \varphi(u_t)$ and u_t and R_t satisfy:

$$R_t + \pi(Q_t - 1)R_t = Q_t \varphi'(u_t), \tag{B.6}$$

and

$$R_t = \alpha \left(u_t K_t\right)^{\alpha-1}. \tag{B.7}$$

Equation (B.6) determines the capacity utilization rate u_t. Since φ is convex, we can show that u_t decreases with Q_t, the shadow price of capital. Intuitively, an increase in Q_t leads to an increased marginal cost of depreciated capital more than an increased marginal benefit of more investment.

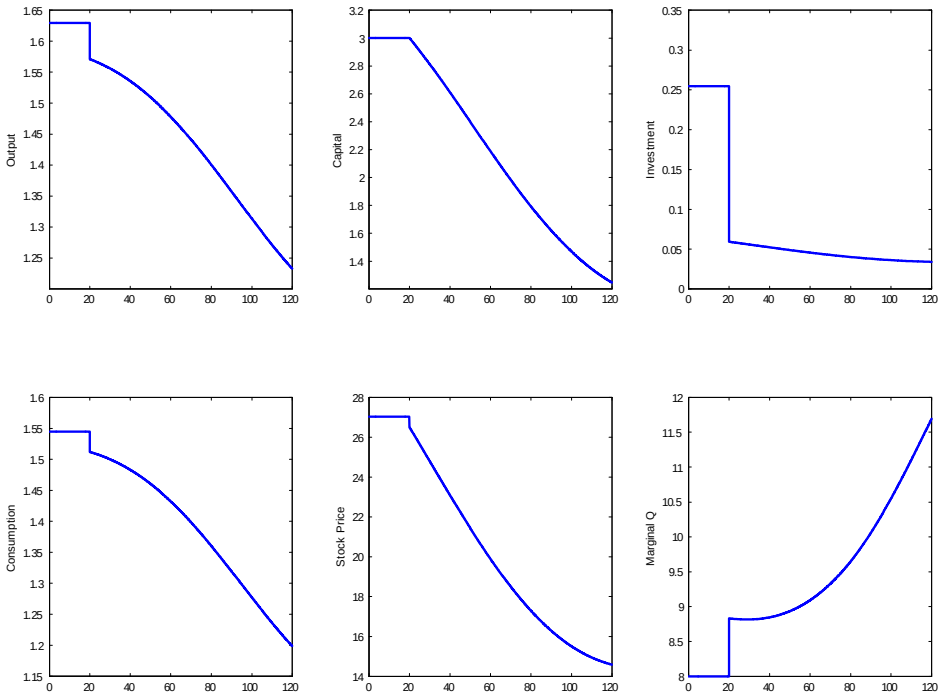

Figure 5: This figure plots the dynamics of the stationary equilibrium with capacity utilization and stochastic bubbles. Set the paramter values as follows: $r = 0.02$, $\alpha = 0.4$, $\pi = 0.01$, $\theta = 0.05$, $\xi = 0.2$, $\gamma = 0.4$, $\delta_0 = 0.0075$, and $\delta_1 = 0.0245$.

Next, we turn to the equilibrium with stochastic bubbles. When the bubble bursts, the economy will be at the bubbleless equilibrium described in the previous proposition. We write the capital price in a feedback form $Q_t = Q^*(K_t)$ for some function K_t. As in Section 6, we characterize the equilibrium before the bubble bursts as follows:

Proposition 11 *Suppose $Q_t > 1$. Before the bubble bursts, the equilibrium with stochastic bubbles (B_t, Q_t, K_t, u_t) satisfies the following system of differential equations:*

$$\dot{Q}_t = (r + \delta_t + \theta)Q_t - \theta Q_t^* - u_t R_t - \pi(Q_t - 1)(u_t R_t + \xi Q_t), \tag{B.8}$$

$$\dot{B}_t = (r + \theta)B_t - \pi(Q_t - 1)B_t, \tag{B.9}$$

$$\dot{K}_t = -\delta_t K_t + (u_t R_t K_t + \xi Q_t K_t + B_t)\pi, \tag{B.10}$$

where u_t and R_t are given by (B.6) and (B.7), and $Q_t^* = Q^*(K_t)$.

As in Section 6, we can simulate the impact of the collapse of bubbles. In doing so, we specify: $\varphi(u_t) = \delta_0 + \delta_1 \frac{u_t^{1+\gamma}}{1+\gamma}$ and take parameter values given in Figure 5. Suppose the economy is in a steady state initially and then the bubble bursts at $t = 20$. On impact, Q_t immediately jumps and so does the capacity utilization rate. Since one unit of installed capital becomes more valuable, the capacity utilization rate decreases to slow down depreciation. As a result, both output and consumption drop immediately. Figure 5 reveals that the economy enters into a prolonged recession after the burst of bubbles.

C Intertemporal Borrowing and Saving

In the base model studied in Section 2, we have considered intratemporal debt and assumed that firms cannot borrow and save intertemporally. In this appendix, we shall relax this assumption. We assume that there is no intratemporal debt and firms can borrow and save intertemporally by trading risk-free one-period private bonds. Households can also trade these bonds, but face short sales constraints. Intuitively, each period firms without investment opportunities have extra funds and may lend to firms with investment opportunities. They may also save their funds and use these funds to finance investments when investment opportunities arrive in the future. We will show that bubbles on firm assets can still exist even though the steady-state interest rate on the private bonds is positive. In addition, our key insights still carry over to this setup.

Consider a discrete-time setup outlined in Section 2 and time is denoted by $t = 0, 1, 2....$ Let $\beta = e^{-r}$. Let R_{ft} denote the gross interest rate of the bonds. Let L_t^h denote households' bond holdings. The short sales constraint is given by $L_t^h \geq 0$ for all t.

Let $V_{1t}\left(K_t^j, L_t^j\right)$ $(V_{0t}(K_t^j, L_t^j))$ denote the stock market value of a typical firm j with (without) investment opportunities when its capital stock and debt at time t are K_t^j and L_t^j, respectively. We suppress the aggregate state variables in the argument. When $L_t^j < 0$, L_t^j means savings. Then $V_{1t}\left(K_t^j, L_t^j\right)$ satisfies the Bellman equation:

$$\begin{aligned} V_{1t}\left(K_t^j, L_t^j\right) &= \max_{I_t^j, L_{t+1}^j} R_t K_t^j - I_t^j + \frac{L_{t+1}^j}{R_{ft}} - L_t^j \\ &\quad + \beta\left[\pi V_{1t+1}\left(K_{t+1}^j, L_{t+1}^j\right) + (1-\pi) V_{0t+1}\left(K_{t+1}^j, L_{t+1}^j\right)\right], \end{aligned} \tag{C.1}$$

subject to

$$K_{t+1}^j = (1-\delta) K_t^j + I_t^j, \tag{C.2}$$

$$D_t^j + I_t^j = R_t K_t^j + \frac{L_{t+1}^j}{R_{ft}} - L_t^j, \tag{C.3}$$

$$D_t^j \geq -(1-\lambda) L_t^j, \tag{C.4}$$

and a borrowing constraint (C.6) described below. Equation (C.3) is a flow of funds constraint, where D_t^j means dividend payments if $D_t^j \geq 0$ and new equity if $D_t^j < 0$. We assume that external equity finance is costly so that the value of new equity is limited by a fraction $(1-\lambda)$ of debt as shown in (C.4). This constraint ensures that new equity is not sufficient to finance investment and firms still need debt financing. If $\lambda = 1$, then firms cannot raise any new equity.

A firm without investment opportunities solves the following dynamic programming problem:

$$\begin{aligned} V_{0t}\left(K_t^j, L_t^j\right) &= \max_{L_{t+1}^j} R_t K_t^j + \frac{L_{t+1}^j}{R_{ft}} - L_t^j \\ &\quad + \beta\left[\pi V_{1t+1}\left((1-\delta) K_t^j, L_{t+1}^j\right) + (1-\pi) V_{0t+1}\left((1-\delta) K_t^j, L_{t+1}^j\right)\right], \end{aligned} \tag{C.5}$$

subject to a borrowing constraint (C.6) described below and a constraint on new equity. We will show below that both constraints will not bind in equilibrium.

We now introduce the credit constraint:

$$\beta E_t V_{t+1}\left(K_{t+1}^j, L_{t+1}^j\right) \geq \beta E_t V_{t+1}\left(K_{t+1}^j, 0\right) - \beta E_t V_{t+1}(\xi K_t^j, 0), \tag{C.6}$$

where we define

$$E_t V_{t+1}\left(K_{t+1}^j, L_t^j\right) = \pi V_{1t+1}\left(K_{t+1}^j, L_{t+1}^j\right) + (1-\pi) V_{0t+1}\left(K_{t+1}^j, L_{t+1}^j\right).$$

We follow the discussion in Section 2 and interpret this borrowing constraint as an incentive constraint in an optimal contract. In any period t, firm j chooses to borrow L_{t+1}^j/R_{ft}. It may default on debt L_{t+1}^j at the beginning of period $t+1$. If it does not default, it obtains continuation value $\beta E_t V_{t+1}\left(K_{t+1}^j, L_{t+1}^j\right)$. If it defaults, debt is renegotiated and the repayment L_{t+1}^j is relieved. The lender seizes the collateralized assets ξK_t^j and keeps the firm running with these assets by reorganizing the firm. Thus, the threat value to the lender is $\beta E_t V_{t+1}(\xi K_t^j, 0)$. Assume that firms have a full bargaining power. Then the expression on the right-hand side of (C.6) is the value to the firm if it chooses to default. Thus, the constraint in (C.6) ensures that firm j does not have an incentive to default.

We conjecture that:

$$V_{1t}\left(K_t^j, L_t^j\right) = v_{1t} K_t^j - v_{1t}^L L_t^j + b_{1t} \text{ and } V_{0t}\left(K_t^j, L_t^j\right) = v_{0t} K_t^j - v_{0t}^L L_t^j + b_{0t}, \tag{C.7}$$

where v_{1t}, v_{1t}^L, b_{1t}, v_{0t}, v_{0t}^L, and b_{0t} are variables to be determined. Define the following notations: $v_t = \pi v_{1t} + (1-\pi) v_{0t}$, $v_t^L = \pi v_{1t}^L + (1-\pi) v_{0t}^L$, $b_t = \pi b_{1t} + (1-\pi) b_{0t}$, $Q_t = \beta v_{t+1}$, $B_t = \beta b_{t+1}$, and $Q_t^L = \beta v_{t+1}^L$. Using the preceding conjecture and the notations, we rewrite the credit constraint (C.6) as:

$$Q_t^L L_{t+1}^j \leq Q_t \xi K_t^j + B_t, \tag{C.8}$$

where Q_t^L may be interpreted as the shadow price of loans.

Turn to the characterization of equilibrium. We shall show that in equilibrium firms with investment opportunities choose to borrow and firms without investment opportunities choose to save and lend. In addition, households are borrowing constrained so that they do not hold any bonds.

We start with problem (C.5) for a firm without investment opportunities. If this firm chooses to save, then the borrowing constraint (C.8) does not bind. Substituting conjecture (C.7) into (C.5) yields.

$$v_{0t} K_t^j - v_{0t}^L L_t^j + b_{0t} = \max_{L_{t+1}^j} \; R_t K_t^j + \frac{L_{t+1}^j}{R_{ft}} - L_t^j + Q_t (1-\delta) K_t^j - Q_t^L L_{t+1}^j + B_t. \tag{C.9}$$

The following conditions must hold for an interior solution:

$$\frac{1}{R_{ft}} = Q_t^L, \tag{C.10}$$

and $v_{0t} = R_t + (1-\delta)Q_t$, $b_{0t} = B_t$ and $v^L_{0t} = 1$.

Next, we consider problem (C.1) for a firm with investment opportunities. Substituting conjecture (C.7) into this problem yields:

$$\begin{aligned} v_{1t}K^j_t - v^L_{1t}L^j_t + b_{1t} &= \max_{I^j_t, L^j_{t+1}} R_t K^j_t - I^j_t + \frac{L^j_{t+1}}{R_{ft}} - L^j_t \\ &\quad + Q_t[(1-\delta)K^j_t + I^j_t] - Q^L_t L^j_{t+1} + B_t. \end{aligned} \tag{C.11}$$

If $Q_t > 1$ then both the borrowing constraint (C.8) and the constraint (C.4) must bind. Using these constraints as well as (C.10) and (C.3), we obtain:

$$I^j_t = R_t K^j_t + Q_t \xi K^j_t + B_t - \lambda L^j_t. \tag{C.12}$$

Substituting this equation back into the Bellman equation (C.11) and matching coefficients, we obtain:

$$v^L_{1t} = 1 + \lambda(Q_t - 1), \tag{C.13}$$

$$v_{1t} = R_t + (1-\delta)Q_t + (Q_t - 1)(\xi Q_t + R_t), \tag{C.14}$$

$$b_{1t} = B_t + (Q_t - 1)B_t. \tag{C.15}$$

By definition of B_t and Q_t introduced in this appendix, we can show that:

$$B_t = \beta B_{t+1}(1 + \pi(Q_{t+1} - 1), \tag{C.16}$$

$$Q_t = \beta[R_{t+1} + (1-\delta)Q_{t+1} + \pi(Q_{t+1} - 1)(\xi Q_{t+1} + R_{t+1})]. \tag{C.17}$$

The aggregate capital stock satisfies:

$$\begin{aligned} K_{t+1} &= (1-\delta)K_t + \pi(R_t K_t + Q_t \xi K_t + B_t) - \pi\lambda \int L^j_t dj \\ &= (1-\delta)K_t + \pi(R_t K_t + Q_t \xi K_t + B_t), \end{aligned} \tag{C.18}$$

where the second line follows from the bond market-clearing condition $\int L^j_t dj = 0$, when households do not hold any bonds.

By definition of Q^L_t, $v^L_{0t} = 1$, and equations (C.10) and (C.14), we can show that:

$$\frac{1}{R_{ft}} = Q^L_t = \beta\pi v^L_{1t} + (1-\pi)\, v^L_{0t} = \beta[1 + \pi\lambda(Q_{t+1} - 1)]. \tag{C.19}$$

If $Q_{t+1} > 1$, then

$$\beta R_{ft} = \frac{1}{1 + \pi\lambda(Q_{t+1} - 1)} < 1. \tag{C.20}$$

Thus, households prefer to sell bonds until their borrowing constraints bind so that $L_t^h = 0$ for all t. Because firms with investment opportunities choose to borrow and invest, for the bond market to clear, firms without investment opportunities must save and lend. By (C.16) and (C.20), we find that $R_{ft} > B_{t+1}/B_t$ for $\lambda \in (0,1)$ and $B_t > 0$ so that the interest rate of bonds is larger than the growth rate of bubbles. Only when $\lambda = 1$, $R_{ft} = B_{t+1}/B_t$.

Notice that equations (C.16)-(C.18) are exactly the same as equations (16)-(18) for $dt = 1$ and $r = -\log(\beta)$. Thus, our analyses of bubbleless and bubbly equilibria in Sections 4-5 can be carried over to the present setup, by suitably translating the continuous-time results into the discrete time results.

In the steady state, $B_t = B$ is constant over time, where B may be zero or positive. But the steady-state net rate of interest rate on bonds $R_f - 1$ is positive. The steady state aggregate dividends D satisfy:

$$D = RK - \pi I = RK - \pi(RK + Q\xi K + B).$$

By (C.18), the steady state capital stock K satisfies:

$$\delta K = \pi(RK + Q\xi K + B).$$

Combining the above two equations yields $D = (R - \delta) K > 0$, where the inequality follows from (28) or (34). Because firms with investment opportunities borrow and raise new equity to finance investment, firms without investment opportunities must save and payout dividends for the aggregate dividends D to be positive. This result is true in the neighborhood of the steady state. Thus, the constraint on new equity finance for firms without investment opportunities will never bind, confirming our previous claim.

D Idiosyncratic Investment-Specific Shocks

In the main text, we have assumed that investment opportunities arrive stochastically according to a Poisson process. In this appendix, we modify our model by considering the case where firms are subject to idiosyncratic investment-specific shocks with a continuous distribution in a discrete time setup. We will show that our key insights carry over to this case. In addition, we will show that stock-price bubbles can arise even though capital can be fully pledgeable, e.g, $\xi = 1$.

We replace equation (3) with

$$K_{t+1}^j = (1 - \delta) K_t^j + \varepsilon_t^j I_t^j, \tag{D.1}$$

where ε_t^j represents idiosyncratic investment-specific shocks. Assume that ε_t^j is independently and identically distributed over time and across firms. It is drawn from a continuous distribution function Φ with the support $[\varepsilon_{\min}, \varepsilon_{\max}]$, where $\varepsilon_{\min} \geq 0$.

Let $V_t\left(K_t^j, \varepsilon_t^j\right)$ denote the cum-dividends stock value after the idiosyncratic shock ε_t^j is realized. It satisfies the following Bellman equation:

$$V_t(K_t^j, \varepsilon_t^j) = \max_{I_t^j} \; R_t K_t^j - I_t^j + \beta E_t V_{t+1}(K_{t+1}^j, \varepsilon_{t+1}^j), \tag{D.2}$$

subject to the investment constraint (5) and the credit constraint:

$$L_t^j \leq \beta E_t V_{t+1}(\xi K_t^j, \varepsilon_{t+1}^j). \tag{D.3}$$

Here E_t is the conditional expectation operator with respect to ε_{t+1}^j. We conjecture that firm value takes the following form:

$$V_t\left(K_t^j, \varepsilon_t^j\right) = v_t\left(\varepsilon_t^j\right) K_t^j + b_t\left(\varepsilon_t^j\right),$$

where $v_t\left(\varepsilon_t^j\right)$ and $b_t\left(\varepsilon_t^j\right)$ are functions to be determined.

Following a similar proof for Proposition 1, we can show that optimal investment is given by:

$$I_t^j = \begin{cases} 0 & \varepsilon_t^j < 1/Q_t \\ R_t K_t^j + \xi Q_t K_t^j + B_t & \varepsilon_t^j \geq 1/Q_t \end{cases}, \tag{D.4}$$

where

$$B_t = \beta \int b_{t+1}(\varepsilon)\, d\Phi(\varepsilon), \; Q_t = \beta \int v_{t+1}(\varepsilon)\, d\Phi(\varepsilon),$$

$$v_t(\varepsilon_t^j) = \begin{cases} R_t dt + (1 - \delta dt) Q_t & \varepsilon_t^j < 1/Q_t \\ (1 - \delta - \xi) Q_t + (R_t + \xi Q_t) Q_t \varepsilon_t^j & \varepsilon_t^j \geq 1/Q_t \end{cases},$$

$$b_t(\varepsilon_t^j) = \begin{cases} B_t & \varepsilon_t^j \leq 1/Q_t \\ B_t Q_t \varepsilon_t^j & \varepsilon_t^j > 1/Q_t \end{cases}.$$

Clearly, when the investment-specific shock ε_t^j is small enough, the firm chooses to reduce investment as much as possible so that investment reaches the lower bound. When the investment-specific shock ε_t^j is large enough, the firm chooses to increase investment until reaching the upper bound.

Following a similar proof for Proposition 2, we can then derive the equilibrium system. We then take the continuous time limit to obtain the following result.

Proposition 12 *The equilibrium dynamics (B_t, Q_t, K_t) satisfy the following system of difference equations:*

$$B_t = \beta B_{t+1} \int \max(\frac{\varepsilon}{\varepsilon^*_{t+1}}, 1)\phi(\varepsilon)d\varepsilon \tag{D.5}$$

$$Q_t = \beta \left[\alpha K_{t+1}^{\alpha-1} + (1-\delta)Q_{t+1} + \int_{\varepsilon^*_{t+1}} \left(\alpha K_{t+1}^{\alpha-1} + \xi Q_{t+1}\right)\left(Q_{t+1}\varepsilon - 1\right)\phi\left(\varepsilon\right)d\varepsilon \right] \tag{D.6}$$

$$K_{t+1} = (1-\delta)K_t + (\alpha K_t^{\alpha} + K_t \xi Q_t + B_t) \int_{\varepsilon^*_t}^{\infty} \varepsilon\phi(\varepsilon)d\varepsilon \tag{D.7}$$

where

$$\varepsilon^*_t = \frac{1}{Q_t} \tag{D.8}$$

and the transversality condition:

$$\lim_{T\to\infty} \beta^T Q_T K_{T+1} = 0, \quad \lim_{T\to\infty} \beta^T B_{T+1} = 0.$$

Clearly, there is one equilibrium without bubbles in which $B_t = 0$ for all t. To study the existence of a bubbly equilibrium, we consider the Pareto distribution $\Phi\left(\varepsilon\right) = 1 - \varepsilon^{-\sigma}$ where $\sigma > 1$ and $\varepsilon \geq 1$. We shall focus on the steady state. In the bubbly steady state, B_t, Q_t and K_t are equal to some constants B, Q_b and K_b over time, respectively. We can then use the above proposition to show that:

$$Q_b = \left[(\sigma-1)\left(\beta^{-1}-1\right)\right]^{\frac{1}{\sigma}},$$

$$R_b = Q_b[1 - \beta(1-\delta) + (\beta-1)\xi] > 0,$$

$$\begin{aligned} \frac{B}{K_b} &= \frac{\delta}{\int_{\varepsilon^*_b}^{\infty} \varepsilon\phi(\varepsilon)d\varepsilon} - R_b - \xi Q_b \\ &= Q_b \left(\frac{\delta}{\sigma\left(\beta^{-1}-1\right)} - 1 + \beta\left(1-\delta\right) - \beta\xi \right). \end{aligned}$$

For $B > 0$, we need

$$\xi < \frac{\delta}{\sigma\left(1-\beta\right)} + 1 - \frac{1}{\beta} - \delta \equiv \bar{\xi}.$$

This is the condition for the existence of bubbles for the model in this appendix. We can see that if

$$1 < \sigma < \frac{\delta}{\left(\delta + \beta^{-1}\right)\left(1-\beta\right)},$$

then $\bar{\xi} > 1$. Thus, bubbles can arise even though the firm pledge all its assets as collateral (i.e., $\xi = 1$).

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