Debt Financing in Asset Markets*

ZHIGUO HE WEI XIONG

The widely used short-term debt such as overnight repos and commercial paper is regarded as one of the main sources of instability leading to the distress of many financial institutions during the financial crisis of 2007-2008. What explains the popularity of short-term debt in financing asset market investments? Geanakoplos (2010) presents a dynamic model of the joint equilibrium of asset markets and credit markets, in which optimistic buyers of a risky asset use the asset as collateral to raise debt financing from less optimistic creditors. The asset's collateral value depends on the marginal creditor's asset valuation, and determines the buyers' purchasing capacity to bid up the asset price. Geanakoplos posits that short-term debt allows the asset buyers to maximize riskless leverage.[1] As highlighted by a number of studies (e.g., Acharya, Gale, and Yorulmaker (2011), and He and Xiong (20011a, b)), short-term debt exposes the asset buyers to rollover risk, which is reflected in the fact that the asset buyers in Geanakoplos' model may not be able to obtain refinancing after a negative fundamental shock. As a result, they are forced to transfer their asset holdings to the creditors, causing a leverage cycle to emerge. The presence of such rollover risk prompts a fundamental question regarding the optimality of short-term debt financing.

Moreover, one might argue that as creditors are less optimistic and undervalue risky debt issued by optimistic asset buyers, it is not desirable for the buyers to use risky financing. Following Geanakoplos' framework, Simsek (2011) examines a setting in which asset buyers have to hold their positions until the asset matures. He shows that optimistic asset buyers may use risky debt financing despite their debt being undervalued by the creditors. The

*He: Booth School of Business, University of Chicago (email: zhiguo.he@chicagobooth.edu); Xiong: Department of Economics, Princeton University (email: wxiong@princeton.edu). We thank Dan Cao, Zhenyu Gao, Adriano Rampini, Alp Simsek and seminar participants at 3rd Theory Workshop on Corporate Finance and Financial Markets, Northwestern University, Stockholm School of Economics, and University of Minnesota for helpful discussion and comments.

[1] The existing literature also offers several other advantages of short-term debt. First, short-term debt can act as a disciplinary device by giving the creditors the option to pull out if they discover that firm managers are pursuing value-destroying projects (e.g., Calomiris and Kahn (1991)). Second, the short commitment period also makes short-term debt less information sensitive and thus less exposed to adverse-selection problems (e.g., Gorton and Pennacchi (1990)).

contrast between Simsek's and Geanakoplos' analysis suggests that the asset's tradability in the interim periods may affect the debt financing of asset buyers.

This paper examines the optimal use of debt financing by extending the dynamic framework of Geanakoplos along two dimensions. First, we allow the asset's fundamental value to follow a generic binomial tree, which takes multiple values after multiple periods, as opposed to the two values in Geanakoplos' model. Second, we allow two groups of agents to have time-varying beliefs about the asset's fundamental, and, in particular, to have more dispersed beliefs after a negative fundamental shock. The more dispersed beliefs make debt refinancing more expensive to the optimistic asset buyers and thus expose them to greater rollover risk. The rollover risk motivates long-term debt financing. To uncover the role played by the asset's tradability, we also contrast two settings: the main setting whereby agents can freely trade the asset on any date, and a benchmark setting whereby agents cannot trade after the initial date. The non-tradability is due to the presence of informational and trading frictions that prevent agents from promptly responding to fundamental fluctuations, and makes the benchmark setting essentially static and analogous to that of Simsek.

Our analysis delivers several results. First, to our surprise, the maximum riskless short-term leverage is optimal despite the presence of rollover risk. While long-term debt financing can dominate short-term debt financing when the rollover risk is sufficiently high (i.e., belief dispersion becomes sufficiently high after a negative shock), the improved investment opportunity in this situation makes saving cash initially more desirable than acquiring the position. Thus, long-term debt financing is dominated either by short-term debt financing or by saving cash. This result explains the pervasive use of short-term debt in practice. It also verifies the statement put forward by Geanakoplos, albeit due to optimists' cash saving incentives rather than his argument that short-term debt allows optimists to maximize riskless leverage.

Second, we highlight the role played by the asset's tradability. By making it possible to buy the liquidated collateral on an interim date, the asset's tradability motivates some optimists to save cash, which, in turn, boosts the creditors' valuation of the collateral on the initial date, as well as the optimists' purchasing capacity to bid up the asset price.

Finally, despite the market incompleteness due to the borrowing constraints faced by the optimists, we derive a risk-neutral representation of the equilibrium prices of the asset and debt contracts collateralized by the asset. This is because the payoff of a collateralized debt

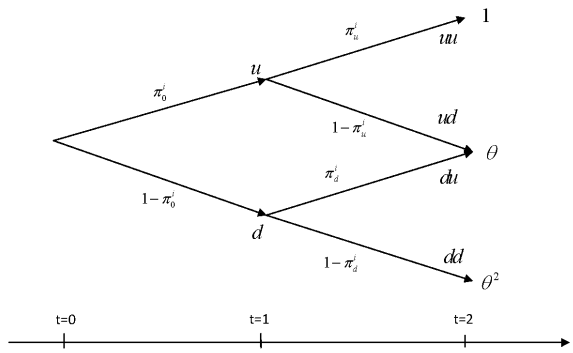

Figure 1: Timeline.

contract is monotonic with respect to the asset's value and thus shares the same marginal investor as the asset. This representation facilitates our analysis and could prove useful in analyzing issues related to collateralized debt financing in more complex settings.

The paper is organized as follows. Section 1 describes the model. We derive the static benchmark setting in Section 2, and analyze the dynamic setting in Section 3. Due to space constraints, our description of the model is brief. Instead, we provide an online appendix to cover more details and all the technical derivations. The online Appendix also extends the two-period model described in the paper to N periods.

1 The Model

Consider a model with three dates and two periods. The date is indexed by $t = 0, 1, 2$. There is a long-term risky asset. The asset's final payoff on date 2 is determined by a publicly observable binomial tree (Figure 1). We specify the asset's final payoff $\widetilde{\theta}$ at the end of the tree path uu as 1, at the end of ud and du as θ, and at the end of dd as θ^2, where $\theta \in (0, 1)$. The probability of the tree going up in each period is unobservable. There are two groups of risk-neutral agents, who differ in their beliefs about these probabilities. We collect an agent's beliefs on the three intermediate states, one on date 0 and two on date 1 (u and d), by $\{\pi_0^i, \pi_u^i, \pi_d^i\}$ where $i \in \{h, l\}$ indicates the agent's type. We assume that the h-type agents are always more optimistic than the l-type agents (the superscript "h" and "l" stand for high and low): $\pi_n^h \geq \pi_n^l$ for any $n \in \{0, u, d\}$. We emphasize that each agent's belief changes over time, driven by either his learning process or sentimental fluctuation. As a result, the belief dispersion between the optimists and pessimists varies over time. Our analysis highlights

the case where the optimists face rollover risk in the intermediate state d when the belief dispersion between the optimists and pessimists diverges (i.e., $\pi_d^h - \pi_d^l \geq \pi_0^h - \pi_0^l \geq 0$).[2]

The pessimists (l-type agents) are initially endowed with all of the asset, normalized to be one unit. On date 0, each optimist (an h-type agent) is endowed with c dollars of cash. Our analysis focuses on the financing of the optimists' asset purchases when c is small. For simplicity, we assume that both the risk-free interest rate and the agents' discount rate are zero and short sales of the asset are not allowed. We also assume that the pessimists are competitive and always demand zero expected return from the asset or any debt contract they hold.

2 The Static Setting

We first consider a benchmark setting in which agents cannot trade the asset on date 1 and are restricted from using debt contracts that mature on date 1. This setting is effectively static with three possible final states on date 2. Like Geanakoplos (2010), we allow the optimists to use their asset holdings as collateral to obtain debt financing. We focus on non-contingent debt contracts, which specify constant debt payments at maturity unless the borrowers default.[3]

Suppose that an optimist uses a debt contract, collateralized by one unit of the asset with a promised payment $F \in \left[\theta^2, \theta\right]$ due on date 2. The debt's eventual payment is $\widetilde{\theta} \wedge F \equiv \min\left(F, \widetilde{\theta}\right)$. A pessimistic creditor will grant the following credit for the contract: $D_0 = \mathbb{E}_0^l\left[\widetilde{\theta} \wedge F\right]$. To establish the asset position, the optimist has to use $p_0 - \mathbb{E}_0^l\left[\widetilde{\theta} \wedge F\right]$ of his own cash. This amount is the so-called haircut. With a cash endowment of c, the optimist can purchase $\frac{c}{p_0 - \mathbb{E}_0^l\left[\widetilde{\theta} \wedge F\right]}$ units of the asset. Since each unit gives an expected payoff of $\mathbb{E}_0^h\left[\widetilde{\theta} - \widetilde{\theta} \wedge F\right]$, he can maximize his expected value by choosing the debt promise F:

$$\max_F \quad c\mathbb{E}_0^h\left[\widetilde{\theta} - \widetilde{\theta} \wedge F\right] / \left\{p_0 - \mathbb{E}_0^l\left[\widetilde{\theta} \wedge F\right]\right\}. \tag{1}$$

In the joint equilibrium of the asset and credit markets, each optimist maximizes the

[2] We simplify the continuum of constant beliefs considered by Geanakoplos (2009) to two types. By focusing on homogeneous optimists, we can isolate an optimist's incentive to save cash from a pure belief effect.

[3] Non-contingent debt contracts are widely used in practice. Townsend (1979) explains their popularity based on the argument that non-contingent debt contracts circumvent the cost of verifying the value of the collateral as long as the borrower makes the promised payment. Diamond (1984) and Bolton and Scharfstein (1990) also derive the optimality of non-contingent debt based on unobservability of cash flows.

objective in (1). As an optimist could also choose to acquire the debt issued by other optimists instead of the asset, another equilibrium condition requires that the marginal value of investing one dollar in the asset is as good as investing in debt issued by other optimists. Furthermore, the market clearing condition requires that the asset price is determined by the optimists' aggregate purchasing power (the sum of their cash endowment c and the borrowed credit $\mathbb{E}_0^l\left[\widetilde{\theta} \wedge F^*\right]$) if the price is between the pessimists' and optimists' asset valuations. In the online Appendix, we describe these conditions in detail and derive the equilibrium. Here, we highlight the key characteristics of the equilibrium.

The equilibrium crucially depends on the belief dispersion between the optimists and pessimists about the highest state uu being higher than their belief dispersion about the three upper states $\{uu, ud, du\}$:

$$\frac{\pi_0^h \pi_u^h}{\pi_0^l \pi_u^l} \geq \frac{\pi_0^h + \pi_d^h - \pi_0^h \pi_d^h}{\pi_0^l + \pi_d^l - \pi_0^l \pi_d^l}. \tag{2}$$

This condition implies that the belief dispersion is concentrated in the highest state and is consistent with the monotonic belief ordering imposed by Simsek (2011). As we will discuss later in the dynamic setting, this condition also implies that the rollover risk faced by the optimists with short-term debt financing in state d of date 1 is modest relative to their initial speculative incentives on date 0.

Under the monotone belief dispersion specified in (2), the optimists will use risky debt with a promised payment θ to finance their asset acquisition when their cash endowment is sufficiently low (Proposition 2 of the online Appendix). Despite their debt being undervalued by the creditors, the optimists believe that the asset price is sufficiently attractive to offset the high financing cost. This result is consistent with that of Simsek (2011), and builds on the three-point distribution of the asset's fundamental on the final date, as opposed to the two-point distribution in Geanakoplos (2010). On the other hand, if (2) is not satisfied, the optimists use only riskless debt with a promised payment θ^2 to finance their asset acquisition (Proposition 3 of the online Appendix).

3 The Dynamic Setting

We now allow agents to trade the risky asset on the interim date—date 1—after the two possible states, u and d, are revealed to the public. As a result, each optimist can also make endogenous investment and leverage decisions in these two interim states. Moreover, each

optimist has the choice to use either short-term debt maturing on date 1 or long-term debt maturing on date 2 to finance his asset position on date 0. We allow the collateralized debt contracts (either long-term or short-term) to be tradable on dates 0 and 1. The marginal investor of these debt contracts in each of the states is determined by the equilibrium—his identity can be either optimist or pessimist and can be different across states.

3.1 Risk-Neutral Representation of Prices

We first establish a risk-neutral representation of the prices of the asset and debt contracts collateralized by the asset. This representation is reminiscent of the standard no-arbitrage risk-neutral price representation in complete markets, even though the borrowing constraints faced by the optimists make the markets incomplete. Instead, this representation builds on the idea that the payoff of a collateralized debt contract is monotonic with respect to the asset's value and, as a result, the debt shares the same marginal investor as the asset.

Given the scarcity of cash to the optimists, cash allows them to earn extra rents. Their marginal value of cash in a state reflects the investment opportunity in the state and acts as a link between the prices of the asset and collateralized debt contracts. Consider the intermediate state d. The asset price p_d must be between the optimists' and pessimists' asset valuations: $p_d \in \left[\mathbb{E}_d^l\left(\widetilde{\theta}\right), \mathbb{E}_d^h\left(\widetilde{\theta}\right)\right]$. One dollar of cash allows an optimist to acquire the asset at a discount to his valuation. He can use one dollar, levered up by using a debt contract collateralized by one unit of the asset and with a promise of θ^2 (the maximum riskless promise), to acquire a position of $\frac{1}{p_d-\theta^2}$ units of the asset. This position gives an expected gross profit of

$$v_d = \frac{\pi_d^h\left(\theta-\theta^2\right)}{p_d-\theta^2}$$

which, as proved in Proposition 4 of the online Appendix, is exactly his marginal value of cash in the state. As $p_d \in \left[\mathbb{E}_d^l\left(\widetilde{\theta}\right), \mathbb{E}_d^h\left(\widetilde{\theta}\right)\right]$, $v_d \in \left[1, \pi_d^h/\pi_d^l\right]$. Similarly, his marginal value of cash in state u is $v_u = \frac{\pi_u^h(1-\theta)}{p_u-\theta}$. On date 0, the optimists can either use the cash, levered up by collateralized debt, to acquire the asset or save it for date 1. Thus, his marginal value of cash is the maximum of the two choices:

$$v_0 = \max\left(u_0, \pi_0^h v_u + \left(1-\pi_0^h\right) v_d\right), \text{ where } u_0 = \frac{\pi_0^h v_u\left(p_u-p_d\right)}{p_0-p_d}.$$

We can use the optimist's marginal value of cash to represent the risk-neutral beliefs of

the marginal investor in an intermediate state:

$$\phi_s \equiv \frac{\pi_s^h}{v_s} \in (0,1)\,, \text{ where } s = u \text{ or } d, \text{ and } \phi_0 \equiv \max\left(\frac{\pi_0^h v_u}{v_0}, \pi_0^l\right) \in (0,1)\,.$$

By construction, the price of the asset can be represented by the marginal investor's risk-neutral beliefs. More subtle is that the price of any collateralized debt contract must follow the same representation to avoid "arbitrage" to the same marginal investor (Proposition 6 of the online Appendix):

$$P_u = \phi_u P_{uu} + (1-\phi_u) P_{ud}, \quad P_d = \phi_d P_{ud} + (1-\phi_d) P_{dd}, \quad P_0 = \phi_0 P_u + (1-\phi_0) P_d,$$

where P_s denotes the price of the asset or a collateralized debt contract in state s.

3.2 An Optimist's Problem

An individual optimist faces three alternatives on date 0. He can establish an asset position by using short-term debt financing; he can use long-term debt financing; or he can simply save cash for establishing a position later on date 1.

We first study an optimistic buyer's financing decision in establishing an initial asset position on date 0. Suppose that he uses a debt contract $\widetilde{F}$ (either long-term or short-term) collateralized by each unit of asset. As we discussed, the amount of credit he obtains by selling the debt to the marginal investor in the market is

$$D_0\left(\widetilde{F}\right) = \phi_0 D_u\left(\widetilde{F}\right) + (1-\phi_0) D_d\left(\widetilde{F}\right),$$

where we denote D_s the debt value in state s. Each dollar of cash allows him to establish a position of $x = \frac{1}{p_0 - D_0\left(\widetilde{F}\right)}$ units of the asset. This position gives an expected profit of

$$\begin{aligned} V_0\left(\widetilde{F}\right) &= x\left[\pi_0^h v_u\left(p_u - D_u\left(\widetilde{F}\right)\right) + \left(1-\pi_0^h\right) v_d\left(p_d - D_d\left(\widetilde{F}\right)\right)\right] \\ &= \frac{\left(\pi_0^h v_u p_u + \left(1-\pi_0^h\right) v_d p_d\right) - \left(\pi_0^h v_u D_u\left(\widetilde{F}\right) + \left(1-\pi_0^h\right) v_d D_d\left(\widetilde{F}\right)\right)}{p_0 - D_0\left(\widetilde{F}\right)}. \end{aligned} \tag{3}$$

This equation reflects two effects. One is a leverage effect: by using a more aggressive debt contract, the optimist can obtain more credit and thus establish a greater position, as shown by the denominator of the formula. The other is a debt-cost effect: a more aggressive debt contract also implies a higher expected debt payment in the future, as shown by the term $\left(\pi_0^h v_u D_u\left(\widetilde{F}\right) + \left(1-\pi_0^h\right) v_d D_d\left(\widetilde{F}\right)\right)$ in the numerator.

Debt Maturity Choice We use the optimist's expected value from an asset position in (3) to compare a pair of long-term and short-term debt contracts, which have different promises (F_L and F_S) but give the same initial credit. From the optimist's perspective, he prefers the one with a lower expected debt cost. For these two contracts to give the same initial credit, we must have

$$\phi_0 D_u^S + (1-\phi_0) D_d^S = \phi_0 D_u^L (F_L) + (1-\phi_0) D_d^L (F_L),$$

where, with potential abuse of notation, D_s^L denotes the value of the long-term debt in state $s \in \{u, d\}$. We define the optimist's cash-value-adjusted probability as

$$\widehat{\pi}_0^h \equiv \frac{\pi_0^h v_u}{\pi_0^h v_u + \left(1-\pi_0^h\right) v_d}.$$

We can show that the short-term contract gives the lower expected debt cost to the optimistic borrower if and only if $\widehat{\pi}_0^h \geq \phi_0$ (Proposition 7 of the online Appendix).

This result establishes that the optimist's debt maturity choice depends on his cash-value-adjusted belief relative to the creditor's belief. The basic intuition works as follows: refinancing the short-term debt allows the borrower to trade a higher payment in state d for a lower payment in state u. Whether this trade is preferable depends on whether the borrower's belief about the probability of the up state, after adjusting for his marginal value of cash in the future states, is higher than the creditor's belief. This result shows that long-term debt could dominate short-term debt in financing the optimists' asset positions, and thus contrasts the standard intuition that optimists always prefer short-term debt financing if they have to borrow from pessimistic creditors.

Cash Saving The optimist can also choose to save cash for the next date. Interestingly, saving cash dominates over using long-term debt to finance an asset position if $\widehat{\pi}_0^h < \phi_0$ (Proposition 8 of the online Appendix). This is because the asset appears to be overvalued to the optimist after adjusting for his marginal value of cash on the following date. This result shows that when long-term debt dominates short-term debt for financing the optimist's asset position, saving cash dominates for establishing the position. Thus, the optimist will never strictly prefer long-term debt in the equilibrium. He could be indifferent between using long-term and short-term debt in financing his position only if $\widehat{\pi}_0^h = \phi_0$.

Optimal Debt Financing In summary, if $\widehat{\pi}_0^h > \phi_0$, the asset price is attractive to the optimist and he establishes an asset position by using short-term debt with the maximum

riskless promise to pay p_d on date 1; if $\widehat{\pi}_0^h = \phi_0$, he is indifferent between saving cash and establishing the position.

3.3 The Equilibrium

As the pessimists are initially endowed with all of the asset, they are able to finance any asset purchase of the optimists. Due to their risk neutrality and deep pockets, their asset valuations put lower bounds on the prices of the asset and any collateralized debt contract. On date 0, the pessimists are the marginal investor if and only if $\phi_0 = \pi_0^l$. They are indifferent between selling and retaining the asset in this case, and prefer selling if $\phi_0 > \pi_0^l$.

The joint equilibrium of the asset and credit markets can be characterized by the prices of the asset on date 0 and in states u and d of date 1: $\{p_0, p_u, p_d\}$, which also determine the price of any collateralized debt contract based on the risk-neutral representation we discussed earlier; the fraction of optimists who establish asset positions on date 0: $\alpha \in [0,1]$; and the fraction of pessimists who sell their asset endowments on date 0: $\lambda \in [0,1]$. In this equilibrium, each optimist and each pessimist solve his optimal decisions based on our earlier discussion, and the markets for the asset and the debt used by optimists must clear. In particular, if the optimists are the marginal investor in the markets, their aggregate purchasing power needs to be sufficient to support the asset price. We present the specific equilibrium conditions for each intermediate state $s \in \{0, u, d\}$ in the online Appendix, which also characterizes various cases that arise in the equilibrium.

Here, we provide a numerical illustration to contrast the dynamic setting with the static benchmark setting. We use a set of baseline parameter values:

$$\theta = 0.4, c = 0.15, \pi_0^h = 0.6, \pi_0^l = 0.55, \pi_u^h = \pi_u^l = 0.5, \pi_d^h = 0.75, \pi_d^l = 0.4. \tag{4}$$

The specified belief structure does not satisfy the monotone belief dispersion given in (2).

Figure 2 illustrates the effects of varying optimists' cash endowment c from 0.42 to 0.07. Panel A depicts the date-0 asset price p_0. The two horizontal lines at the levels of 0.5 and 0.556 represent the pessimists' and the optimists' expectation of the asset's fundamental value, i.e., $\mathbb{E}_0^l\left[\widetilde{\theta}\right]$ and $\mathbb{E}_0^h\left[\widetilde{\theta}\right]$. The equilibrium price has to fall between these two benchmark levels. The dotted-and-dashed line is the asset price under the static setting discussed in Section 2. When c is above 0.4, the equilibrium price is equal to the optimists' valuation. Once c falls below 0.4, the asset price starts to fall with c and is determined by the optimists'

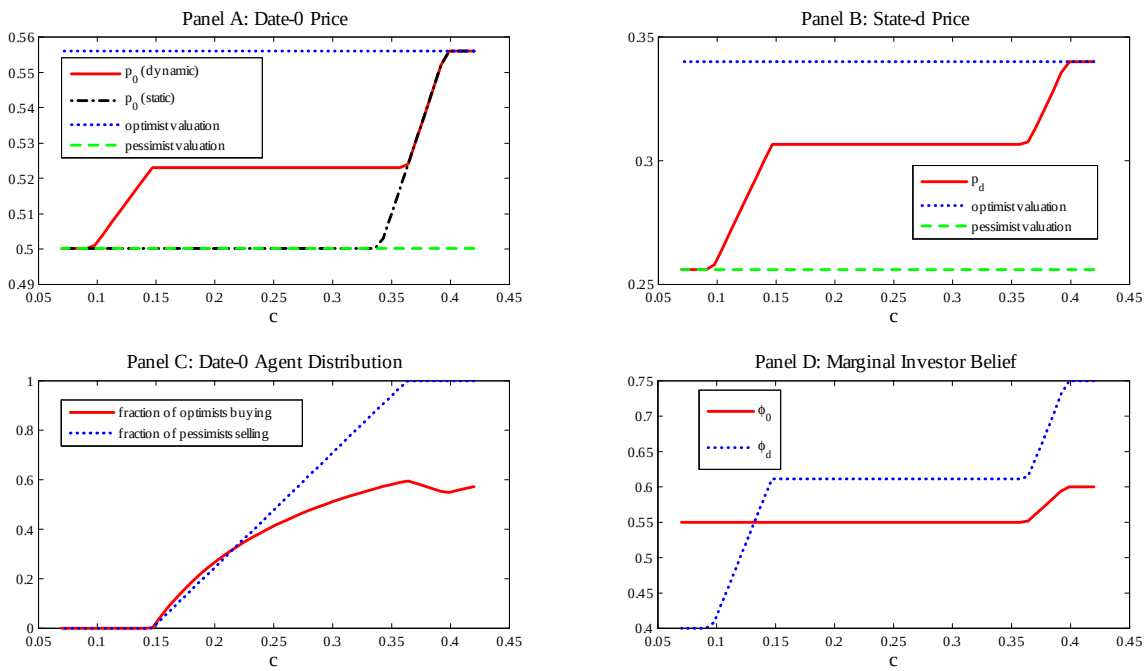

Figure 2: The effects of optimists' cash endowments based on the baseline parameters given in (4).

purchasing power rather than their valuation. As c falls below 0.336, the marginal investor shifts to the pessimists and the asset price is equal to the pessimists' valuation.

The solid line in Panel A gives the asset price in the dynamic setting discussed in this section. Like the dotted-and-dashed line, when c is above 0.4, the equilibrium price is equal to the optimists' valuation. When c falls below 0.4, the asset price starts to fall with c because the optimists cannot afford to support the price at their valuation. Interestingly, as c drops below 0.357 but above 0.14, the price levels off at the level 0.523. In this range, the asset price is substantially higher than that in the static setting. Panel C reveals the key source of this difference—as c falls, a greater fraction of the optimists $(1-\alpha)$ choose to save cash on date 0 and more pessimists hold their asset endowments (λ). As further shown by Panels B and D, in this range the optimists' cash saving sustains the asset price p_d and the implied belief of the marginal investor ϕ_d in state d of date 1 at constant levels, and, in particular, ensures the optimists as the marginal investor in this state (i.e., $\phi_d > \pi_d^l$). Panel D also shows that the marginal investor on date 0 is the pessimists ($\phi_0 = \pi_0^l = 0.55$). Despite this, the date-0 asset price p_0 is higher than that in the static setting because the pessimists anticipate that p_d will be determined by the optimists.

As c gets lower than 0.14 but higher than 0.09, all optimists save cash on date 0 ($\alpha = 0$)

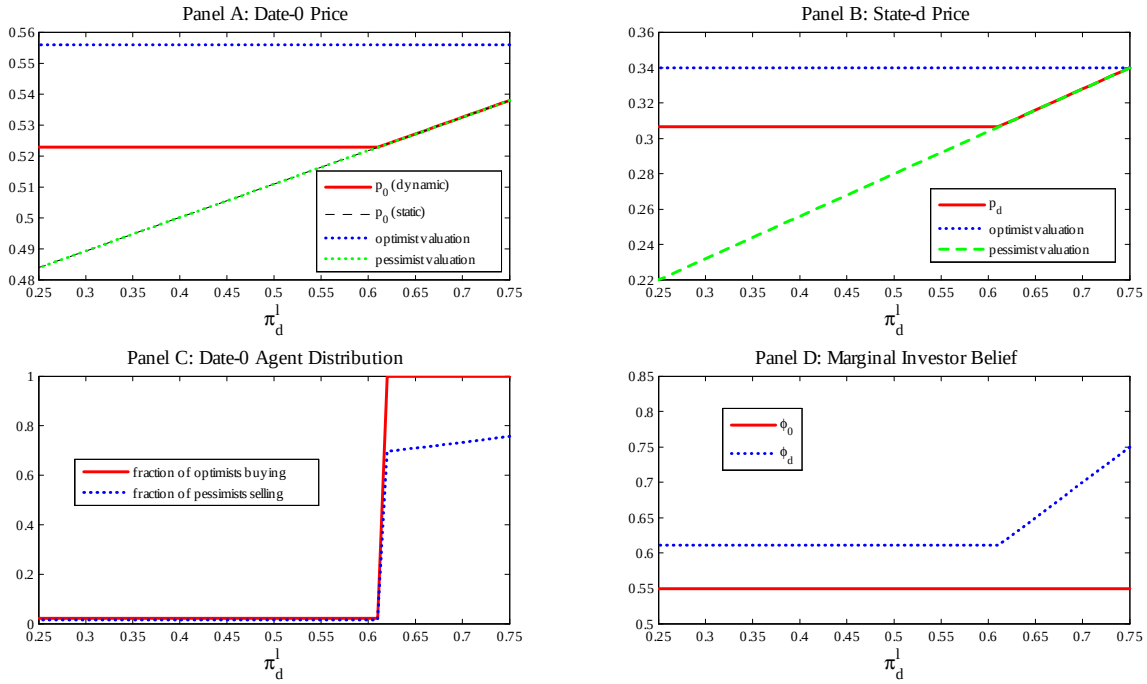

Figure 3: The effects of belief dispersion in state d based on the baseline parameters given in (4).

and all pessimists hold their asset endowments ($\lambda = 0$). In this case, the optimists remain the marginal investor of the asset in state d. As the optimists have saved all of their cash endowments to state d, the asset price in this state now falls with c, which in turn causes the asset price on date 0 to fall with c as well. Nevertheless, the date-0 price p_0 is higher than that in the static setting because the pessimists, the marginal investor on date 0, anticipate selling their asset holdings to optimists in state d. Only when c falls below 0.09 do the pessimists become the marginal investor of the asset both on date 0 and in state d of date 1. As a result, the prices in the dynamic and static settings coincide.

Taken together, Figure 2 demonstrates a significant impact of the asset's tradability when the optimists' cash endowment c is in the intermediate range between 0.09 and 0.357. In this range, the tradability induces at least some optimists to preserve cash from date 0 to state d, where the marginal value of cash is highest. These optimists' cash saving supports the asset price in state d, which in turn motivates the pessimists to assign a higher collateral value to the asset on date 0.

Figure 3 illustrates the effects of the belief dispersion between the optimists and pessimists in state d of date 1. In particular, we fix optimists' cash endowment c at its baseline value and

vary π_d^l—the pessimists' belief in state d, from 0.75 to 0.25. As π_d^l drops, the belief dispersion between the optimists and pessimists ($\pi_d^h - \pi_d^l$) widens. The widened belief dispersion can make refinancing more costly to the optimists in this state if they choose to use short-term debt on date 0. Panel D indicates that when π_d^l is above 0.61, the marginal investor is the pessimists as $\phi_0 = \pi_0^l$ and $\phi_d = \pi_d^l$. As a result, both date-0 price p_0 and state-d price p_d fall as π_d^l falls (Panels A and B). Interestingly, as π_d^l falls below 0.61, the optimists start to save cash on date 0 (Panel C). As a result, the marginal investor in state d shifts to the optimists with the implied belief ϕ_d fixed at a constant level 0.61 (panel D) and the asset price p_d fixed at 0.307 (panel B). This in turn allows the pessimists, the marginal investor of the asset on date 0, to value the asset p_0 at a constant level 0.523. Taken together, Figure 3 shows that cash saving by some optimists on date 0, induced by the better opportunity in state d, can alleviate the rollover risk faced by other optimists who choose to acquire asset positions on date 0 by using short-term debt.

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Appendix of "Debt Financing in Asset Markets"

ZHIGUO HE WEI XIONG

This online appendix provides a more detailed description of the model discussed in the main paper, and lists all of the propositions and proofs. The appendix also extends the model to N periods. We describe the static setting in Section 1 and the dynamic setting in Section 2. Section 3 extends the model. Section 4 gives the proofs of all propositions and lemmas.

1 The Static Setting

In equilibrium, the date-0 price of the asset $p_0 \in \left[\mathbb{E}_0^l\left[\widetilde{\theta}\right], \mathbb{E}_0^h\left[\widetilde{\theta}\right]\right)$. Suppose that an optimist uses a debt contract, collateralized by one unit of the asset with a promised payment $F \in \left[\theta^2, \theta\right]$ due on date 2. The debt's eventual payment is $\widetilde{\theta} \wedge F \equiv \min\left(F, \widetilde{\theta}\right)$. A pessimistic creditor will grant the following credit for the contract: $D_0 = \mathbb{E}_0^l\left[\widetilde{\theta} \wedge F\right]$. To establish the asset position, the optimist has to use $p_0 - \mathbb{E}_0^l\left[\widetilde{\theta} \wedge F\right]$ of his own cash. This amount is the so-called haircut. With a cash endowment of c, the optimist can purchase $\frac{c}{p_0 - \mathbb{E}_0^l\left[\widetilde{\theta} \wedge F\right]}$ units of the asset. Since each unit gives an expected payoff of $\mathbb{E}_0^h\left[\widetilde{\theta} - \widetilde{\theta} \wedge F\right]$, he can maximize his expected value by choosing the debt promise F:

$$\max_F c \frac{\mathbb{E}_0^h\left[\widetilde{\theta} - \widetilde{\theta} \wedge F\right]}{p_0 - \mathbb{E}_0^l\left[\widetilde{\theta} \wedge F\right]}. \tag{1}$$

In formulating this objective, we implicitly assume that the credit offered by the pessimistic creditor is the same as the market value of the debt contract. As other optimists may find the debt contract attractive and offer a higher price than pessimists, we need to verify that this is not the case in equilibrium. In fact, this condition does bind in certain situations. Formally, suppose that in equilibrium the optimal debt promise is F^*, which

maximizes the optimist's objective in (1). This implies that the marginal value of one dollar, v_0, to an optimist from acquiring the levered asset position is $\frac{\mathbb{E}_0^h\left[\widetilde{\theta}-\widetilde{\theta}\wedge F^*\right]}{p_0-\mathbb{E}_0^l\left[\widetilde{\theta}\wedge F^*\right]}$.We need to verify that this is not lower than his expected value from using an optimal leverage to acquire the debt contract (which has a payoff of $\widetilde{\theta}\wedge F^*$):

$$\frac{\mathbb{E}_0^h\left[\widetilde{\theta}-\widetilde{\theta}\wedge F^*\right]}{p_0-\mathbb{E}_0^l\left[\widetilde{\theta}\wedge F^*\right]}\geq\max_F\frac{\mathbb{E}_0^h\left[\widetilde{\theta}\wedge F^*-\left(\widetilde{\theta}\wedge F^*\right)\wedge F\right]}{\mathbb{E}_0^l\left[\widetilde{\theta}\wedge F^*\right]-\mathbb{E}_0^l\left[\left(\widetilde{\theta}\wedge F^*\right)\wedge F\right]}. \tag{2}$$

Thus, we call a pair of (p_0, F^*) the joint equilibrium of the asset and credit markets if the following two conditions hold:

1. Given the asset price p_0, F^* maximizes an individual optimist's investment objective in (1) and satisfies the constraint in (2).

2. The asset market clears:

$$p_0=\begin{cases}\mathbb{E}_0^l\left[\widetilde{\theta}\right] & \text{if } c+\mathbb{E}_0^l\left[\widetilde{\theta}\wedge F^*\right]<\mathbb{E}_0^l\left[\widetilde{\theta}\right]\\ c+\mathbb{E}_0^l\left[\widetilde{\theta}\wedge F^*\right] & \text{if } c+\mathbb{E}_0^l\left[\widetilde{\theta}\wedge F^*\right]\in\left[\mathbb{E}_0^l\left[\widetilde{\theta}\right],\mathbb{E}_0^h\left[\widetilde{\theta}\right]\right]\\ \mathbb{E}_0^h\left[\widetilde{\theta}\right] & \text{if } c+\mathbb{E}_0^l\left[\widetilde{\theta}\wedge F^*\right]>\mathbb{E}_0^h\left[\widetilde{\theta}\right]\end{cases}. \tag{3}$$

The market clearing condition requires that if the optimists' aggregate purchasing power, which is determined by the sum of their cash endowment c and the borrowed credit $\mathbb{E}_0^l\left[\widetilde{\theta}\wedge F^*\right]$, is lower than the pessimists' asset valuation $\mathbb{E}_0^l\left[\widetilde{\theta}\right]$, the pessimists will determine the price at their valuation; if the optimists' purchasing power is between the pessimists' and optimists' asset valuations, the asset price p_0 is exactly equal to the optimists' purchasing power; and if the optimists' purchasing power is higher than the optimists' asset valuation $\mathbb{E}_0^h\left[\widetilde{\theta}\right]$, they will bid up the price to their valuation.

The following lemma shows that the optimal debt promise must be between θ^2 and θ.

Lemma 1 *The optimist's optimization problem in (1) implies that* $F^*\in\left[\theta^2,\theta\right]$.

1.1 Monotonic Belief Dispersion

We first consider the situation that the belief dispersion between optimists and pessimists about the highest state uu is higher than their belief dispersion about the three upper states $\{uu, ud, du\}$:

$$\frac{\pi_0^h\pi_u^h}{\pi_0^l\pi_u^l}\geq\frac{\pi_0^h+\pi_d^h-\pi_0^h\pi_d^h}{\pi_0^l+\pi_d^l-\pi_0^l\pi_d^l}. \tag{4}$$

This condition implies that the belief dispersion is concentrated in the highest state and is consistent with the monotonic belief ordering imposed by Simsek (2011). As we will discuss later in the dynamic setting, this condition also implies that the rollover risk faced by optimists with short-term debt financing in state d of date 1 is modest relative to their initial speculative incentives on date 0.

We can prove that in this case, the constraint in (2) always holds in equilibrium. Therefore, the equilibrium can be derived from solving an individual optimist's problem in (1) joint with the market clearing condition (3).

Specifically, define

$$P_M \equiv \left(\pi_0^l + \pi_d^l - \pi_0^l \pi_d^l\right) \frac{\pi_0^h \pi_u^h + \left(\pi_0^h \left(1 - \pi_u^h\right) + \left(1 - \pi_0^h\right) \pi_d^h\right) \theta}{\pi_0^h + \pi_d^h - \pi_0^h \pi_d^h} + \left(1 - \pi_0^l\right) \left(1 - \pi_d^l\right) \theta^2$$

as a critical price level, which under the belief condition (4), satisfies

$$P_M \geq \mathbb{E}_0^l \left[\widetilde{\theta}\right].$$

Then, an optimist's optimal debt promise depends on the asset price:

$$F^*\left(p_0\right) = \begin{cases} \theta^2, & \text{if } p_0 \in \left(P_M, \mathbb{E}_0^h \left[\widetilde{\theta}\right]\right); \\ \text{any value in } \left[\theta^2, \theta\right], & \text{if } p_0 = P_M; \\ \theta, & \text{if } p_0 \in \left(\mathbb{E}_0^l \left[\widetilde{\theta}\right], P_M\right]. \end{cases}$$

By using this debt contract, the optimist obtains a credit of

$$C\left(F^*\right) \equiv \left(1 - \pi_0^l\right) \left(1 - \pi_d^l\right) \theta^2 + \left(\pi_0^l + \pi_d^l - \pi_0^l \pi_d^l\right) F^*.$$

Finally, by imposing the market clearing condition (3), we obtain the equilibrium described in the following proposition.

Proposition 2 *Under the belief condition (4), the asset and credit market equilibrium is characterized by the following four cases:*

1. *if* $c \in \left(P_M - \theta^2, \mathbb{E}_0^h \left[\widetilde{\theta}\right] - \theta^2\right]$, *then* $F^* = \theta^2$ *and* $p_0 = \theta^2 + c$;
2. *if* $c \in \left(P_M - C\left(\theta\right), P_M - \theta^2\right]$, *then* $F^* = \frac{p_M - c - \left(1 - \pi_0^l\right)\left(1 - \pi_d^l\right)\theta^2}{\pi_0^l + \pi_d^l - \pi_0^l \pi_d^l}$ *and* $p_0 = P_M$;
3. *if* $c \in \left(\mathbb{E}_0^l \left[\widetilde{\theta}\right] - C\left(\theta\right), P_M - C\left(\theta\right)\right]$, *then* $F^* = \theta$ *and* $p_0 = C\left(\theta\right) + c$;
4. *if* $c \leq \mathbb{E}_0^l \left[\widetilde{\theta}\right] - C\left(\theta\right)$, *then* $F^* = \theta$ *and* $p_0 = \mathbb{E}_0^l \left[\widetilde{\theta}\right]$.

This proposition shows that when the optimists' initial cash endowment is low (e.g., cases 2, 3, and 4), they will use risky debt to borrow from pessimists to finance their asset positions, despite their debt being undervalued. This is because their speculative incentives are sufficiently strong. In other words, they believe that the asset is more undervalued relative to the debt they use to finance their asset acquisition. This result is consistent with that of Simsek (2011). It is important to note that this result builds on the three-point distribution of the asset's fundamental on date 2, which contrasts the two-point distribution assumed by Geanakoplos (2010).

1.2 Non-monotonic Belief Dispersion

We now consider the situation that the belief dispersion between the optimists and pessimists about the highest state uu is lower than their belief dispersion about the three upper states $\{uu, ud, du\}$:

$$\frac{\pi_0^h \pi_u^h}{\pi_0^l \pi_u^l} < \frac{\pi_0^h + \pi_d^h - \pi_0^h \pi_d^h}{\pi_0^l + \pi_d^l - \pi_0^l \pi_d^l}. \tag{5}$$

Interestingly, under this condition, we can show that there does not exist any equilibrium in which the optimists use risky debt to borrow from the pessimists. This is because the belief condition in (5) implies that the belief dispersion between the optimists and pessimists about the two middle paths (ud and du), which lead to $\widetilde{\theta} = \theta$, is higher than the belief dispersion about the highest state uu, which gives $\widetilde{\theta} = 1$. As a result, if any risky debt, say a debt contract with face value θ is offered in the equilibrium at the pessimists' valuation, then this risky debt offers a strictly better investment opportunity to any optimist than the risky asset itself. Then, some optimists will choose to withdraw from the risky asset to acquire the risky debt, which, in turn, causes optimists' aggregate leverage and the equilibrium asset price to fall. Due to this logic, the constraint in (2) is violated, and the only equilibrium that can satisfy this constraint is the one with only riskless borrowing by optimists from pessimists. We describe the equilibrium in the following proposition

Proposition 3 *Under the belief condition (5), optimists only borrow riskless debt from pessimists and the equilibrium is characterized by three cases:*

1. *if* $c \in \left(\mathbb{E}_0^h\left[\widetilde{\theta}\right] - \theta^2, \mathbb{E}_0^h\left[\widetilde{\theta}\right]\right]$, *then* $F^* = \mathbb{E}_0^h\left[\widetilde{\theta}\right] - c$ *and* $p_0 = \mathbb{E}_0^h\left[\widetilde{\theta}\right]$;
2. *if* $c \in \left(\mathbb{E}_0^l\left[\widetilde{\theta}\right] - \theta^2, \mathbb{E}_0^h\left[\widetilde{\theta}\right] - \theta^2\right]$, *then* $F^* = \theta^2$ *and* $p_0 = \theta^2 + c$;

3. if $c \leq \mathbb{E}_0^l\left[\widetilde{\theta}\right] - \theta^2$, *then* $F^* = \theta^2$ *and* $p_0 = \mathbb{E}_0^l\left[\widetilde{\theta}\right]$.

Taken together, propositions 2 and 3 demonstrate that in the static setting, the debt financing used by the optimists crucially depends on the distribution of the optimists' and pessimists' beliefs across different states. In particular, proposition 2 shows that the optimists may choose to use risky debt financing despite their debt being undervalued by the creditors.

2 Dynamic Setting

In the dynamic setting, agents can trade the risky asset on the interim date—date 1—after the two possible states, u and d, are revealed to the public. As a result, each optimist can also make endogenous investment and leverage decisions in these two interim states. Moreover, each optimist has the choice to use either short-term debt maturing on date 1, or long-term debt maturing on date 2, to finance his asset position on date 0.

We denote the asset price on date 0 and in states u and d of date 1 by $\{p_0, p_u, p_d\}$. We allow all debt contracts (either long-term or short-term) to be tradable on dates 0 and 1. For a long-term debt contract collateralized by one unit of asset with a promise to pay F on date 2, we denote its market valuation on date 0 and in states u and d of date 0 by $\left\{D_0^L(F), D_u^L(F), D_s^L(F)\right\}$, where the superscript L refers to long-term debt and the subscript refers to the relevant state. For a short-term debt contract with a promise to pay F on the following date, we denote its market value on date 0 and in states u and d of date 0 by $\left\{D_0^S(F), D_u^S(F), D_s^S(F)\right\}$. Note that a long-term debt contract that matures on date 2 becomes identical to a short-term debt contract on date 1. Thus, $D_s^L(F) = D_s^S(F)$ for $s \in \{u, d\}$. Because of the presence of heterogeneous agents in the market, it is important to bear in mind that the marginal investor of these debt contracts in each of the states is determined by the equilibrium—his identity can be either optimist or pessimist and can be different across the states.

2.1 Risk-Neutral Representation of Prices

Before deriving each agent's optimization problem, we first establish a risk-neutral representation of the prices of the risky asset and debt contracts collateralized by the asset. This representation is reminiscent of the standard no-arbitrage risk-neutral price representation in complete markets. Due to the borrowing constraints faced by optimists, the markets in our

setting are incomplete even though the asset fundamental follows a binomial tree. Instead, this representation builds on the idea that the payoff of a debt contract collateralized by the asset is monotonic with respect to the asset's value and, as a result, it shares the same marginal investor as the asset.

2.1.1 Optimists' Marginal Value of Cash

Given the scarcity of cash to the optimists, cash allows them to earn extra rents. To illustrate the idea, consider state d of date 1, one period before the risky asset's liquidation value is revealed. Suppose that the asset price is p_d, which lies between the optimists' and pessimists' asset valuations: $p_d \in \left[\mathbb{E}_d^l\left(\widetilde{\theta}\right), \mathbb{E}_d^h\left(\widetilde{\theta}\right)\right]$, where $\mathbb{E}_d^i$ denotes agent-i's conditional expectation in the state. One dollar of cash allows an optimist to acquire the asset at a discount to his valuation. Thus, its marginal value is higher than 1. More specifically, he can use the cash, levered up by using a debt contract collateralized by one unit of asset and with a promise of θ^2 (the maximum riskless promise), to acquire a position of $\frac{1}{p_d-\theta^2}$ units of the asset. This position gives an expected gross profit of $\frac{\pi_d^h\left(\theta-\theta^2\right)}{p_d-\theta^2}$. This expected profit represents his marginal value of cash in state d, as formally derived in the following proposition.[1]

Proposition 4 *Suppose that in state d the market price of the asset $p_d \in \left[\mathbb{E}_d^l\left(\widetilde{\theta}\right), \mathbb{E}_d^h\left(\widetilde{\theta}\right)\right]$. Then, the optimal strategy of an optimist with one dollar of cash is to acquire $\frac{1}{p_d-\theta^2}$ units of the asset by using a debt contract with a promise of θ^2. As a result, his marginal value of cash is*

$$v_d = \frac{\pi_d^h\left(\theta-\theta^2\right)}{p_d-\theta^2}. \tag{6}$$

As the asset price p_d is bounded from below by pessimists' asset valuation $\mathbb{E}_d^l\left(\widetilde{\theta}\right) = \pi_d^l\theta + \left(1-\pi_d^l\right)\theta^2$, it is direct to see that the optimist's marginal value of cash is bounded from above: $v_d \leq \pi_d^h/\pi_d^l$.

We can also establish a similar result for state u. Suppose that in state u the market price of the asset is $p_u \in \left[\mathbb{E}_u^l\left(\widetilde{\theta}\right), \mathbb{E}_u^h\left(\widetilde{\theta}\right)\right]$ and that the marginal investor of the debt contracts

[1] This proposition confirms the basic intuition of Geanakoplos (2009) that in a static binomial tree model, an optimist always prefers using the maximum riskless leverage to finance his asset position. It is also useful to note that if the prices of the asset and the collateralized debt contract are determined by the same marginal investor, the optimist is indifferent to any promise between θ^2 and θ. This is because in this case the cost of the risky part of the debt exactly offsets the expected profit from the asset.

in the markets is pessimists. Then, his marginal value of cash is

$$v_u = \frac{\pi_u^h (1-\theta)}{p_u - \theta}. \tag{7}$$

If the beliefs of optimists and pessimists converge in this state, then the asset price has to be equal to their asset valuations and thus the optimist's marginal value of cash is 1, i.e., $v_u = 1$.

Determining the optimist's marginal value of cash on date 0 is more elaborate. On one hand, he can use the cash, levered up by collateralized debt, to acquire the asset. The marginal value of cash depends on the optimal debt financing. As we will show later, the optimal financing is to use the maximum riskless short-term debt, which gives the optimist a marginal value of

$$u_0 = \frac{\pi_0^h v_u (p_u - p_d)}{p_0 - p_d}. \tag{8}$$

One the other hand, he can save the cash for the next date, which gives him an expected value of $\pi_0^h v_u + \left(1 - \pi_0^h\right) v_d$. Optimizing over these two possible choices gives the optimist's marginal value of cash on date 0:

$$v_0 = \max\left(u_0, \pi_0^h v_u + \left(1 - \pi_0^h\right) v_d\right). \tag{9}$$

2.1.2 Asset Price

Equipped with the optimists' marginal value of cash, the following proposition uses it to establish a risk-neutral representation of the price of the risky asset.

Proposition 5 *Define*

$$\phi_s \equiv \frac{\pi_s^h}{v_s} \in (0,1), \text{ and } \phi_0 \equiv \max\left(\frac{\pi_0^h v_u}{v_0}, \pi_0^l\right) \in (0,1). \tag{10}$$

Then, we have

$$p_u = \phi_u + (1-\phi_u)\theta, \tag{11}$$

$$p_d = \phi_d \theta + (1-\phi_d)\theta^2, \tag{12}$$

$$p_0 = \phi_0 p_u + (1-\phi_0) p_d. \tag{13}$$

Intuitively, ϕ_0, ϕ_u, and ϕ_d reflect the equivalent belief the marginal investor uses to value the asset. Note that the marginal investor can be either an optimist or pessimist. In

particular, equation (13) show that on date 0 whether the marginal investor is an optimist or pessimist depends on the optimists' cash-value-adjusted belief, $\frac{\pi_0^h v_u}{\max\left(u_0, \pi_0^h v_u + \left(1-\pi_0^h\right) v_d\right)}$, relative to the pessimists' belief, π_0^l. Interestingly, a higher expected cash value on date 1, i.e., higher $\pi_0^h v_u + \left(1 - \pi_0^h\right) v_d$, can induce the optimists to save cash and, as a result, cause them to have a lower valuation for the asset on date 0 than the pessimists despite that $\pi_0^h \geq \pi_0^l$.

2.1.3 Debt Values

Note that collateralized debt contracts are contingent claims whose value monotonically increase with the value of the asset. Given the binomial uncertainty across each period, the return of a debt contract reflects the same risk as the asset. Thus, its valuation has to be in line with the asset, i.e., to share the same marginal investor.

Proposition 6 *In state $s \in \{u, d\}$ of date 1, the market price of a collateralized debt contract with a promise to pay F on date 2 is*

$$D_s^S(F) = \mathbb{E}_s^{\phi_s}\left[F \wedge \widetilde{\theta}\right], \tag{14}$$

where $\mathbb{E}_s^{\phi_s}$ denotes the expectation of an agent who believes that the tree will go up with a probability of ϕ_s in the following period. On date 0, the price of a short-term debt contract with a promise to pay F on date 1 is given by

$$D_0^S(F) = \phi_0 (F \wedge p_u) + (1 - \phi_0)(F \wedge p_d),$$

and the price of a long-term debt with a promise to pay F on date 2 is given by

$$D_0^L(F) = \phi_0 D_u^S(F) + (1 - \phi_0) D_d^S(F),$$

where $D_u^S(F)$ and $D_d^S(F)$ are given in (14).

To understand this proposition, consider a risky debt contract in state d with a promise to pay F on date 2. Its payoff is either $F \wedge \theta$ or $F \wedge \theta^2$ on date 2. If $D_d^S(F) < \phi_d (F \wedge \theta) + (1 - \phi_d)\left(F \wedge \theta^2\right)$, then this debt contract offers a higher expected return than the asset to its marginal investor and thus would motivate him to withdraw from the asset to invest in the contract. This cannot occur in equilibrium. Neither can $D_d^S(F) > \phi_d (F \wedge \theta) + (1 - \phi_d)\left(F \wedge \theta^2\right)$, as this valuation is too high for anyone to pay.

2.2 An Optimist's Problem

An individual optimist faces three alternatives on date 0. He can establish an asset position by using his initial cash endowment and short-term debt collateralized by the asset position (we call this strategy S); or by using his initial cash endowment and collateralized long-term debt (we call this strategy L); or he can simply save cash for establishing a position later on date 1 (we call this strategy C). Because the optimist is risk neutral, it is without loss of generality to focus on cornered policies for the optimist's optimization problem.

2.2.1 Optimal Debt Financing

We first study the optimal financing decision for an optimistic buyer to establish an initial asset position on date 0. For each dollar of cash he has, suppose that he establishes a position of x units of the asset by using a debt contract $\widetilde{F}$ collateralized by each unit of asset (the debt can be either long-term or short-term). Proposition 6 gives the amount of credit the optimist can obtain by selling the debt to the marginal investor in the market:

$$D_0\left(\widetilde{F}\right) = \phi_0 D_u\left(\widetilde{F}\right) + (1-\phi_0) D_d\left(\widetilde{F}\right),$$

where we omit the maturity indicator S or L on debt function D. His budget constraint implies that

$$x = \frac{1}{p_0 - D_0\left(\widetilde{F}\right)}. \tag{15}$$

With this position, his expected value from per unit of cash is

$$\begin{aligned} V_0\left(\widetilde{F}\right) &= x\left[\pi_0^h v_u\left(p_u - D_u\left(\widetilde{F}\right)\right) + \left(1-\pi_0^h\right) v_d\left(p_d - D_d\left(\widetilde{F}\right)\right)\right] \\ &= \frac{\left(\pi_0^h v_u p_u + \left(1-\pi_0^h\right) v_d p_d\right) - \left(\pi_0^h v_u D_u\left(\widetilde{F}\right) + \left(1-\pi_0^h\right) v_d D_d\left(\widetilde{F}\right)\right)}{p_0 - D_0\left(\widetilde{F}\right)}. \end{aligned} \tag{16}$$

This equation reflects two effects. One is a leverage effect: by using a more aggressive debt contract, the optimist can obtain more credit and thus establish a greater position, as shown by the denominator of the formula. The other is a debt-cost effect: a more aggressive debt contract also implies a higher expected debt payment in the future, as shown by the term $\left(\pi_0^h v_u D_u\left(\widetilde{F}\right) + \left(1-\pi_0^h\right) v_d D_d\left(\widetilde{F}\right)\right)$ in the numerator.[2]

[2] Irrespective of the debt maturity, we can conveniently write the optimist's per-asset profit in state s of

2.2.2 Debt Maturity Choice

Given the optimist's expected value from establishing an asset position in (16), we now compare a pair of long-term and short-term debt contracts, which have different promises (F_L and F_S) but give the same initial credit. From the optimist's perspective, he prefers the one with a lower expected debt cost. For these two contracts to give the same initial credit, we must have

$$\phi_0 D_u^S + (1-\phi_0) D_d^S = \phi_0 D_u^L (F_L) + (1-\phi_0) D_d^L (F_L), \tag{17}$$

where, with a bit of abuse of notation, D_s^S (with $s = u$ or d) denotes the value of the short-term debt contract in state s of date 1.

To simplify notation, we define the optimist's cash-value-adjusted probability as

$$\widehat{\pi}_0^h \equiv \frac{\pi_0^h v_u}{\pi_0^h v_u + \left(1-\pi_0^h\right) v_d}. \tag{18}$$

Then, $1-\widehat{\pi}_0^h = \frac{\left(1-\pi_0^h\right)v_d}{\pi_0^h v_u + \left(1-\pi_0^h\right)v_d}$. With these cash-value-adjusted probabilities, it is straightforward to show that the short-term debt is preferable if and only if

$$\widehat{\pi}_0^h D_u^S + \left(1-\widehat{\pi}_0^h\right) D_d^S \leq \widehat{\pi}_0^h D_u^L + \left(1-\widehat{\pi}_0^h\right) D_d^L, \tag{19}$$

which, by taking the difference with (17), is equivalent to

$$\left(\widehat{\pi}_0^h - \phi_0\right)\left[\left(D_u^L - D_d^L\right) - \left(D_u^S - D_d^S\right)\right] \geq 0. \tag{20}$$

We can directly show that the term in the second bracket is positive (i.e., the market value of the long-term debt is more sensitive to the date-1 state than the short-term debt), and thus establish the following proposition.

Proposition 7 *Consider two debt contracts, one short-term and the other long-term, which generate the same date-0 credit. Then, the short-term contract gives the lower expected debt cost to an optimistic borrower if and only if*

$$\widehat{\pi}_0^h \geq \phi_0. \tag{21}$$

date 1 by $p_s - D_s\left(\widetilde{F}\right)$. Then, his date-0 expected profit can be computed based on his belief about the probability of the next-period states adjusted by his marginal values of cash in these states. This argument holds not only for short-term debt, but also for long-term debt. For illustration, consider state u with $F_0^L \in \left(\theta^2, \theta\right)$. The optimist can lever up further by raising $\theta - F_L$ from his one unit of existing asset to purchase $\frac{\theta - F_L}{p_u - \theta}$ additional units of assets, and his total position becomes $1 + \frac{\theta - F_L}{p_u - \theta} = \frac{p_u - F_L}{p_u - \theta}$. As a result, the total value at state u is

$$\frac{p_u - F_L}{p_u - \theta}\pi_u^h (1-\theta) = v_u\left(p_u - D_u\left(\widetilde{F}\right)\right),$$

where we use the definition of v_u in (7) and $D_u\left(\widetilde{F}\right) = F_L$.

This proposition shows that the optimist's debt maturity choice depends on his cash-value-adjusted belief relative to the creditor's belief. The basic intuition works as follows: refinancing of short-term debt allows the borrower to trade a higher payment in the future down state for a lower payment in the up state. This trade explains the insensitiveness of short-term debt to realization of the future state, as paying more in the down state makes the value of the short-term debt less sensitive to the state. Therefore, whether the tradeoff between a higher payment in the down state and a lower payment in the up state is preferable depends on whether the borrower's belief about the probability of the up state is higher than the creditor's belief. As the borrower also needs to consider his marginal value of cash in the future states, his cash-value-adjusted probability is the relevant belief for comparison with that of the creditor.

Proposition 7 shows that long-term debt could dominate short-term debt in financing optimists' asset positions. This result thus contradicts the standard intuition that optimists always prefer short-term debt financing if they have to borrow from pessimistic creditors.

2.2.3 Cash Saving

The optimist can also choose to save cash for the next date. Interestingly, the following proposition shows that saving capital dominates using long-term debt to finance an asset position if (21) is violated.

Proposition 8 *Suppose that $\widehat{\pi}_0^h < \phi_0$. Then, on date 0 the optimist prefers saving cash to establishing an asset position by using long-term debt.*

When $\widehat{\pi}_0^h < \phi_0$, we have

$$p_0 > \widehat{\pi}_0^h p_u + \left(1 - \widehat{\pi}_0^h\right) p_d = \frac{\pi_0^h v_u p_u}{\pi_0^h v_u + \left(1 - \pi_0^h\right) v_d} + \frac{\left(1 - \pi_0^h\right) v_d p_d}{\pi_0^h v_u + \left(1 - \pi_0^h\right) v_d}.$$

We can interpret $\widehat{\pi}_0^h p_u + \left(1 - \widehat{\pi}_0^h\right) p_d$ as the optimist's asset holding value, after taking into account his marginal value of cash in different states of date 1. Then, it is clear that he should not acquire the asset given that the asset price is above his valuation. This proposition provides an interesting result in that when long-term debt dominates short-term debt for financing the optimist's asset position, saving cash dominates over establishing the position. This result together with Proposition 7 implies that the optimist will never strictly prefer long-term debt in the equilibrium. He would be indifferent between using long-term and short-term debt in financing his position only if $\widehat{\pi}_0^h = \phi_0$.

2.2.4 Leverage Choice

If $\widehat{\pi}_0^h > \phi_0$, we have $p_0 < \widehat{\pi}_0^h p_u + \left(1 - \widehat{\pi}_0^h\right) p_d$, i.e., the asset price is below the optimist's asset valuation. Then, the optimist will find it optimal to establish an asset position by using short-term debt financing. The following proposition shows that the optimist will always use a riskless debt contract with the maximum promise to pay p_d on date 1. The proposition also shows that he is indifferent between saving cash and establishing the position if $\widehat{\pi}_0^h = \phi_0$.

Proposition 9 *When $\widehat{\pi}_0^h > \phi_0$, it is optimal for the optimist to establish an asset position by using a short-term debt contract with a promise to pay p_d on date 1. When $\widehat{\pi}_0^h = \phi_0$, the optimist is indifferent between saving cash and establishing the position.*

Taken together, Propositions 7, 8, and 9 demonstrate that an individual optimist chooses either to save cash for the next period or to acquire an asset position financed by using the maximum riskless short-term debt. The only exception to this result is that when $\widehat{\pi}_0^h = \phi_0$, he can be indifferent between using long-term or short-term debt and between using risky or riskless debt. This is because in this situation, the asset and any debt contract are fairly priced from his perspective.[3]

The result that short-term debt is the only form of debt financing in the equilibrium is useful for several reasons. First, it explains the dominance of short-term debt usage by financial institutions to finance their asset positions. Second, it justifies a common practice in dynamic asset pricing models in which agents' debt maturity choice in financing their investment positions is ignored. Our model demonstrates that in a general binomial setting it is without loss of generality to focus on short-term debt rather than long-term debt, which can substantially complicate the equilibrium analysis.

It is important to note that the driving force for optimists' preference for using short-term debt in our model is different from the argument put forth by Geanakoplos (2009). He argues that short-term debt allows an optimist to maximize riskless leverage. In contrast, our analysis shows that the key difference between short-term and long-term debt is refinancing, which allows the borrower to swap debt payment from the interim down state to the interim up state. As a result, the borrower prefers long-term debt exactly when he highly values payoff in the interim down state. But, in this situation, he should just save cash to take

[3] As he is indifferent between different debt contracts in this situation, we will assume that he chooses riskless short-term debt in our later analysis of the equilibrium. We can verify that allowing the optimist to use other contracts in this situation does not affect the asset price in the equilibrium.

advantage of the improved investment opportunity in this state. Our result thus builds on optimists' ability to trade the risky asset on the interim date and the resulting incentive to save cash. In this regard, our model highlights the importance of the asset's tradability on the interim date in affecting the optimists' financing decision.

2.3 A Pessimist's Problem

As the pessimists are initially endowed with all of the asset, they are able to finance any asset purchase of the optimists. Due to their risk neutrality and deep pockets, their asset valuations put lower bounds on the prices of the asset and any collateralized debt contract. As we discussed earlier, the optimists always use riskless short-term debt to finance their positions. The market price of the debt contract is always the same as the pessimists' valuation. However, the market price of the risky asset may not always reflect the pessimists' valuation. On date 0, a pessimist is indifferent between selling or holding his asset endowment if and only if $\phi_0 = \pi_0^l$. Denote $\lambda \in [0,1]$ as the fraction of the pessimists who sell their asset endowments on date 0. Then

$$\lambda = \begin{cases} 1, & \text{if } \phi_0 > \pi_0^l; \\ \text{any value in } [0,1], & \text{if } \phi_0 = \pi_0^l. \end{cases} \tag{22}$$

2.4 Equilibrium

2.4.1 General Characterization

The joint equilibrium of the asset and credit markets can be characterized by the prices of the asset on date 0 and in states u and d of date 1: $\{p_0, p_u, p_d\}$, which also determine the price of any collateralized debt contract based on the risk-neutral representation derived in Proposition 6; the fraction of the optimists who establish asset positions on date 0: $\alpha \in [0,1]$; and the fraction of the pessimists who sell their asset endowments on date 0: $\lambda \in [0,1]$.

State u of date 1 For a given asset price p_u, Proposition 5 implies that the implied belief of the marginal investor is

$$\phi_u = \frac{p_u - \theta}{1 - \theta}.$$

In equilibrium, $\phi_u \in \left[\pi_u^l, \pi_u^h\right]$. Among the optimists, $(1-\alpha)$ fraction of them have chosen to save cash on date 0 and thus have an aggregate cash of $(1-\alpha)\,c$; the other α fraction have chosen to acquire in aggregate λ units of the asset by using a one-period debt contract with a

promise of p_d and thus have a net worth of $\lambda(p_u - p_d)$. Based on our earlier analysis, in this state any optimist will choose to establish an asset position financed by the maximum debt with a promise of θ. In order for these optimists to buy out the total 1 unit of the asset, their total purchasing power, $\lambda(p_u - p_d) + (1-\alpha)c + \theta$, needs to be above the required purchasing price, p_u. If it falls short of this price, the pessimists will fill in only if $\phi_u = \pi_u^l$. Thus, the market clearing condition in state u is

$$\begin{cases} p_u \geq \lambda(p_u - p_d) + \theta + (1-\alpha)c & \text{if } \phi_u = \pi_u^l \\ p_u = \lambda(p_u - p_d) + \theta + (1-\alpha)c & \text{if } \phi_u \in \left(\pi_u^l, \pi_u^h\right) \\ p_u \leq \lambda(p_u - p_d) + \theta + (1-\alpha)c & \text{if } \phi_u = \pi_u^h \end{cases}. \tag{23}$$

State d of date 1 For a given asset price p_d, Proposition 5 implies that the marginal investor's implied belief is

$$\phi_d = \frac{p_d - \theta^2}{\theta - \theta^2}.$$

In equilibrium, $\phi_d \in \left[\pi_d^l, \pi_d^h\right]$. Those optimists who have taken asset positions financed by the maximum riskless debt are now wiped out and those who choose to save cash on date 0 have an aggregate cash of $(1-\alpha)c$. Then, any optimist with cash will acquire the asset financed by the maximum debt with a promise of θ^2. In order for these optimists to buy out the total 1 unit of the asset, their total purchasing power, $(1-\alpha)c + \theta^2$, needs to be above the required purchasing price, p_d. If it falls short of this price, the pessimists will fill in only if $\phi_d = \pi_d^l$. Thus, the market clearing condition in state d is

$$\begin{cases} p_d \geq (1-\alpha)c + \theta^2 & \text{if } \phi_d = \pi_d^l \\ p_d = (1-\alpha)c + \theta^2 & \text{if } \phi_d \in \left(\pi_d^l, \pi_d^h\right) \\ p_d \leq (1-\alpha)c + \theta^2 & \text{if } \phi_d = \pi_d^h \end{cases}. \tag{24}$$

Date 0 For a given asset price p_0, the marginal investor's implied belief is

$$\phi_0 = \frac{p_0 - p_d}{p_u - p_d}.$$

In equilibrium, $\phi_0 \in \left[\pi_0^l, \widehat{\pi}_0^h\right]$, where $\widehat{\pi}_0^h$ is an optimist's cash-value-adjusted belief on date 0 given by equation (18). Proposition 9 implies that each optimist prefers to establish an asset position financed by a one-period debt contract with a promise of p_d if $\phi_0 < \widehat{\pi}_0^h$; and is indifferent between establishing a position or saving cash if $\phi_0 = \widehat{\pi}_0^h$. Thus, each optimist's optimization leads to the following condition:

$$\begin{cases} \alpha = 1 & \text{if } \phi_0 < \widehat{\pi}_0^h \\ \alpha \in (0,1) & \text{if } \phi_0 = \widehat{\pi}_0^h \end{cases}. \tag{25}$$

As we discussed before, each pessimist will choose to hold his asset endowment only if $\phi_0 = \pi_0^l$, and thus leading to the following condition:

$$\begin{cases} \lambda = 1 & \text{if } \phi_0 > \pi_0^l \\ \lambda \in [0,1] & \text{if } \phi_0 = \pi_0^l \end{cases}. \tag{26}$$

Finally, the market clearing condition requires that the optimists who choose to acquire asset positions are able to finance their asset purchases:

$$\lambda p_0 = \alpha c + \lambda p_d, \tag{27}$$

where λp_0 is the total market value of the λ units of asset sold by the pessimists and $\alpha c + \lambda p_d$ is the total purchasing power of the α fraction of the optimists who have an aggregate cash of αc and are able to obtain an aggregate credit of λp_d by using one-period debt contract with a promise of p_d.

Taken together, equations (23), (24), (25), (26), and (27) allow us to determine the five unknowns: p_0, p_d, p_u, α, and λ.

2.4.2 The Case with Non-monotonic Belief Dispersion

To illustrate the role of the secondary market trading on the equilibrium, we focus on analyzing the case in which the optimists' and pessimists' beliefs diverge from each other in state d of date 1. More specifically, we suppose that

$$\pi_u^h = \pi_u^l, \ \frac{\pi_0^h}{\pi_0^l} < \frac{\pi_0^h + \left(1 - \pi_0^h\right) \pi_d^h}{\pi_0^l + \left(1 - \pi_0^l\right) \pi_d^l}. \tag{28}$$

These conditions imply the non-monotonic belief structure we analyzed in the static setting. The inequality condition also implies that

$$\frac{\pi_0^h}{\pi_0^l} < \frac{\left(1 - \pi_0^h\right) \pi_d^h}{\left(1 - \pi_0^l\right) \pi_d^l}, \tag{29}$$

i.e., the optimists and pessimists disagree more about the second-period asset fundamental conditional on a negative shock in the first period than about the first-period fundamental. The greater belief dispersion in state d implies that it is more costly for an optimist to refinance his risky debt from pessimists in this state, as well as a better investment opportunity if the asset price is determined by the pessimists.

In the benchmark static setting, where the asset is not tradable on the interim date, Proposition 3 shows that the optimists will only use debt contracts with a promise of θ^2

to finance their asset positions and that the equilibrium asset price p_0, if between the pessimists' and optimists' expected asset payoff, is determined by the optimists' aggregate cash endowment c plus θ^2, i.e., $p_0 = c + \theta^2$.

When the optimists have the choice to trade the asset on the interim date, this option creates incentives to save cash on date 0 to take advantage of the better investment opportunity in state d. Cash saving by some optimists boosts the asset price in state d, which, in turn, makes it easier for other optimists to finance their initial asset positions by collateralizing their positions. This feedback mechanism further boosts the asset price on date 0. In particular, we will show that this feedback mechanism can make the price and collateral value of the asset in the dynamic setting higher than those in the benchmark static setting.

We now derive the equilibrium. Given the convergence of the optimists' and pessimists' beliefs in state u, the equilibrium in this state is simple:

$$p_u = \mathbb{E}_u^h\left[\widetilde{\theta}\right] = \mathbb{E}_u^l\left[\widetilde{\theta}\right], \ v_u = 1, \text{ and } \phi_u = \pi_u^h = \pi_u^l.$$

Furthermore, the following lemma establishes that there are always some optimists saving cash on date 0 in the equilibrium.

Lemma 10 *Under the conditions in (28), there is always a fraction of the optimists saving cash on date* 0*, i.e.,* $\alpha < 1$.

In analyzing the equilibrium, we focus on two key variables, ϕ_0 and ϕ_d. They summarize the asset prices on date 0 and in state d of date 0, and determine both the optimists' and pessimists' investment decisions. Note that since it is always optimal for the optimists to save cash on date 0, $v_0 = \pi_0^h + \left(1 - \pi_0^h\right) v_d$, and thus

$$\phi_0 = \max\left(\frac{\pi_0^h}{\pi_0^h + \left(1 - \pi_0^h\right) v_d}, \pi_0^l\right).$$

We describe the equilibrium in the following proposition.

Proposition 11 *Under the conditions in (28), the equilibrium is characterized by the following five cases classified by the level of the optimists' aggregate cash endowment* c *going from high to low:*

1. $c \geq \mathbb{E}_0^h\left[\widetilde{\theta}\right] - \theta^2$. *In this case, the optimists have sufficient cash to bid up the asset price to their expected asset payoff in state* d *of date* 1*:*

$$\phi_d = \pi_d^h \text{ and } p_d = \mathbb{E}_d^h\left[\widetilde{\theta}\right]$$

and on date 0*:*

$$\phi_0 = \pi_0^h \text{ and } p_0 = \mathbb{E}_0^h\left[\widetilde{\theta}\right].$$

The pessimists all sell their asset holdings on date 0*,* $\lambda = 1$*; only a fraction of the optimists need to purchase the asset using one-period debt with a promise of* $\mathbb{E}_d^h\left[\widetilde{\theta}\right]$*,* $\alpha = \left(\mathbb{E}_0^h\left[\widetilde{\theta}\right] - \mathbb{E}_d^h\left[\widetilde{\theta}\right]\right)/c$*.*

2. $c \in \left[\frac{\pi_0^l}{\pi_0^h}\left[\pi_0^h\left(p_u - \theta^2\right) + \left(1-\pi_0^h\right)\pi_d^h\left(\theta - \theta^2\right)\right], \mathbb{E}_0^h\left[\widetilde{\theta}\right] - \theta^2\right]$*. In this case, the optimists cannot maintain the asset price at their expected future asset payoff, but nevertheless remain as the marginal investor of the asset. They determine the asset price based on the financing they can obtain: in state* d *of date* 1*:*

$$\begin{aligned}
\phi_d &= \frac{c\left(1-\pi_0^h\right)\pi_d^h}{\pi_0^h\left(p_u - c - \theta^2\right) + \left(1-\pi_0^h\right)\pi_d^h\left(\theta-\theta^2\right)} \geq \phi_d^* \equiv \frac{\pi_0^l\left(1-\pi_0^h\right)\pi_d^h}{\left(1-\pi_0^l\right)\pi_0^h} > \pi_d^l, \\
p_d &= \theta^2 + \frac{c\left(\theta-\theta^2\right)\left(1-\pi_0^h\right)\pi_d^h}{\pi_0^h\left(p_u - c - \theta^2\right) + \left(1-\pi_0^h\right)\pi_d^h\left(\theta-\theta^2\right)};
\end{aligned}$$

and on date 0*:*

$$\begin{aligned}
\phi_0 &= \frac{\pi_0^h}{\pi_0^h + \left(1-\pi_0^h\right)\pi_d^h/\phi_d} \geq \pi_0^l, \\
p_0 &= c + \theta^2.
\end{aligned}$$

The pessimists all sell their asset holdings on date 0*,* $\lambda = 1$*; only a fraction of the optimists need to purchase the asset by using one-period debt with a promise of* p_d*,* $\alpha = \left(p_0 - p_d\right)/c$*.*

3. $c \in \left[\frac{\pi_0^l\left(1-\pi_0^h\right)\pi_d^h}{\pi_0^h\left(1-\pi_0^l\right)}\left(\theta-\theta^2\right), \frac{\pi_0^l}{\pi_0^h}\left[\pi_0^h\left(p_u-\theta^2\right) + \left(1-\pi_0^h\right)\pi_d^h\left(\theta-\theta^2\right)\right]\right]$*. On date* 0*, the marginal investor of the asset shifts to the pessimists and optimists become indifferent between acquiring the asset and saving cash. In state* d*, the optimists remain as the marginal investor of the asset. As* c *changes inside the given range, the marginal investor's equivalent belief and the asset price remain at constant levels:*

$$\begin{aligned}
\phi_d &= \phi_d^* \equiv \frac{\pi_0^l\left(1-\pi_0^h\right)\pi_d^h}{\left(1-\pi_0^l\right)\pi_0^h} > \pi_d^l, \\
p_d &= p_d^* \equiv \theta^2 + \frac{\pi_0^l\left(1-\pi_0^h\right)\pi_d^h}{\pi_0^h\left(1-\pi_0^l\right)}\left(\theta-\theta^2\right),
\end{aligned}$$

due to the adjustment of the fraction of the optimists acquiring positions on date 0*:*

$$\alpha = 1 - \frac{p_d^* - \theta^2}{c} \geq 0.$$

As a result, the asset price on date 0 *is also a constant:*

$$\begin{aligned}\phi_0 &= \pi_0^l, \\ p_0 &= p_0^* = \pi_0^l p_u + \left(1 - \pi_0^l\right) p_d^*.\end{aligned}$$

The fraction of the pessimists who sell their asset endowments is $\lambda = \frac{\alpha c}{p_0^* - p_d^*}$.

4. $c \in \left[\pi_d^l \left(\theta - \theta^2\right), \frac{\pi_0^l\left(1-\pi_0^h\right)\pi_d^h}{\pi_0^h\left(1-\pi_0^l\right)} \left(\theta - \theta^2\right)\right]$. *In this case, optimists all choose to save cash* ($\alpha = 0$ *and* $\lambda = 0$) *and manage to maintain the asset price in state* d *at*

$$p_d = c + \theta^2$$

with the implied marginal investor's belief of

$$\phi_d = \frac{c}{\theta - \theta^2} \geq \pi_d^l.$$

Pessimists are the marginal investor of the asset on date 0*:*

$$\begin{aligned}\phi_0 &= \pi_0^l, \\ p_0 &= \pi_0^l p_u + \left(1 - \pi_0^l\right)\left(c + \theta^2\right).\end{aligned}$$

5. $c < \pi_d^l \left(\theta - \theta^2\right)$. *In this case, pessimists become the marginal investor of the asset in both state* d*:*

$$\phi_d = \pi_d^l \text{ and } p_d = \mathbb{E}_d^l\left[\widetilde{\theta}\right],$$

and on date 0*:*

$$\phi_0 = \pi_0^l \text{ and } p_0 = \mathbb{E}_0^l\left[\widetilde{\theta}\right].$$

Optimists all save cash on date 0*:* $\alpha = \lambda = 0$.

Cases 3 and 4 in Proposition 11 highlight the key difference between the dynamic and static settings (Proposition 3)—by saving their limited cash from date 0 to state d where the marginal value of cash is highest, optimists can effectively support the asset price in state d, which in turn induces higher asset collateral value on date 0 and motivates pessimists to value the asset higher.

3 Generalizing the Model to N Periods

This section extends the model to a setting with T periods. The asset's fundamental follows a binomial tree, i.e., given the current state it can either go up or down in the following period. The liquidation value of the asset on the final date (date T) can take $T+1$ possible values, denoted by $j = 1, ..., T+1$ from low to high. On a prior date N, there are $N+1$ possible states. Like before, there are two groups of agents, with one group holding more optimistic beliefs about the tree while the other group holds more pessimistic beliefs. Denote the group-i's ($i \in \{h, l\}$) belief in state j of date n by $\pi^i_{n,j}$ with $\pi^h_{n,j} \geq \pi^l_{n,j}$ for $\forall n \in \{0, 1, ..., T-1\}$ and $\forall j \in \{1, ..., n+1\}$. In addition, suppose that the optimists have limited capital while the pessimists always have sufficient capital.

We can establish the following proposition.

Proposition 12 *Consider a binomial-tree setting with T periods.*

1. *There exists a set of implied beliefs $\left\{\phi_{n,j} \in [0,1]\right\}$ of the marginal investor, which permits a risk-neutral representation of the risky asset and any debt contract collateralized by the asset on each node of the tree.*

2. *Consider the j-th node of date $n < T$. Suppose that $\widehat{\pi}^h_{n,j}$ is an optimist's cash-value-adjusted belief on the node. If $\widehat{\pi}^h_{n,j} > \phi_{n,j}$, then he finds it optimal to acquire an asset position by financing the position by using the maximum riskless one-period debt; if $\widehat{\pi}^h_{n,j} < \phi_{n,j}$, then he prefers saving cash for the next period; and if $\widehat{\pi}^h_{n,j} = \phi_{n,j}$, then he is indifferent between saving cash and acquiring an asset position with any debt financing.*

4 Proofs for Propositions

We formally provide proofs for all of the propositions and lemmas described above.

4.1 Proof of Lemma 1

We first rule out the case of $F^* < \theta^2$. Suppose that this case holds true. Then, the debt contract is risk free across all of the four possible paths, i.e., $\widetilde{D} = F$. As a result, the optimist can obtain a credit of F^* and his expected debt cost is also F^*. Then, his expected value in

(1) becomes

$$\frac{c}{p_0 - F^*}\left[\mathbb{E}_0^h\left(\widetilde{\theta}\right) - F^*\right].$$

Now, consider increasing the debt promise by a tiny amount ϵ. The debt contract is still risk free, and the optimist's expected value becomes

$$\frac{c}{p_0 - F^* - \epsilon}\left[\mathbb{E}_0^h\left(\widetilde{\theta}\right) - F^* - \epsilon\right].$$

Since $p_0 < \mathbb{E}_0^h\left(\widetilde{\theta}\right)$, this expression is increasing with ϵ. In other words, the optimist is better-off by borrowing more. This contradicts with F^* being the optimal debt promise. Thus, the optimal debt promise cannot be lower than θ^2.

Next, suppose that $F^* > \theta$. Since the debt promise is higher than θ, the optimist always defaults on the debt contract except at the end of the path uu. Thus, the optimist's profit from his position is $1 - F^*$ at the end of the path uu, and 0 at the end of the other paths. Then, his expected value is

$$\frac{c}{p_0 - \mathbb{E}_0^l\left(\widetilde{\theta} \wedge F^*\right)}\mathbb{E}_0^h\left(\widetilde{\theta} - \widetilde{\theta} \wedge F^*\right).$$

Consider reducing the debt promise by a small amount ϵ, which only affects the payoff at the end of the path uu. The optimists expected value is now

$$\frac{c\mathbb{E}_0^h\left[\widetilde{\theta} - \widetilde{\theta} \wedge \left(F^* - \widetilde{\theta} \wedge F^*\right)\right]}{p_0 - \mathbb{E}_0^l\left[\widetilde{\theta} \wedge \left(F^* - \widetilde{\theta} \wedge F^*\right)\right]} = \frac{c\left[\mathbb{E}_0^h\left(\widetilde{\theta} - \widetilde{\theta} \wedge F^*\right) + \pi_0^h\pi_u^h\epsilon\right]}{p_0 - \mathbb{E}_0^l\left(\widetilde{\theta} \wedge F^*\right) + \pi_0^l\pi_u^l\epsilon}.$$

This expression is increasing with ϵ if

$$\frac{\mathbb{E}_0^h\left(\widetilde{\theta} - \widetilde{\theta} \wedge F^*\right)}{p_0 - \mathbb{E}_0^l\left(\widetilde{\theta} \wedge F^*\right)} \leq \frac{\pi_0^h\pi_u^h}{\pi_0^l\pi_u^l}.$$

Note that since $p_0 \geq \mathbb{E}_0^l\left(\widetilde{\theta}\right)$, we have

$$\frac{\mathbb{E}_0^h\left(\widetilde{\theta} - \widetilde{\theta} \wedge F^*\right)}{p_0 - \mathbb{E}_0^l\left(\widetilde{\theta} \wedge F^*\right)} \leq \frac{\mathbb{E}_0^h\left(\widetilde{\theta} - \widetilde{\theta} \wedge F^*\right)}{\mathbb{E}_0^l\left(\widetilde{\theta} - \widetilde{\theta} \wedge F^*\right)} = \frac{\pi_0^h\pi_u^h}{\pi_0^l\pi_u^l}.$$

Thus, the optimist's expected value increases with ϵ, which contradicts with F^* being the optimal debt promise. This suggests that the optimal debt promise cannot be higher than θ.

4.2 Proof of Proposition 2

The optimist's date-0 expected value by using a debt contract with promise $F \in \left[\theta^2, \theta\right]$ is

$$\max_F \; c\frac{\pi_0^h\pi_u^h + \left(\pi_0^h\left(1-\pi_u^h\right) + \left(1-\pi_0^h\right)\pi_d^h\right)\theta - \left(\pi_0^h + \pi_d^h - \pi_0^h\pi_d^h\right)F}{p_0 - \left(1-\pi_0^l\right)\left(1-\pi_d^l\right)\theta^2 - \left(\pi_0^l + \pi_d^l - \pi_0^l\pi_d^l\right)F} \tag{30}$$

Direct algebra shows that this objective increases with F if and only if

$$p_0 \leq P_M \equiv \left(\pi_0^l + \pi_d^l - \pi_0^l\pi_d^l\right)\frac{\pi_0^h\pi_u^h + \left(\pi_0^h\left(1-\pi_u^h\right) + \left(1-\pi_0^h\right)\pi_d^h\right)\theta}{\pi_0^h + \pi_d^h - \pi_0^h\pi_d^h} + \left(1-\pi_0^l\right)\left(1-\pi_d^l\right)\theta^2. \tag{31}$$

This implies that the optimist should choose $F^* = \theta^2$ if $p_0 < P_M$; $F^* = \theta$ if $p_0 > P_M$; and be indifferent in using any $F \in \left[\theta^2, \theta\right]$ if $p_0 = P_M$.

It is direct to verify that under the belief condition (4), $P_M \geq \mathbb{E}_0^l\left[\widetilde{\theta}\right]$. This is because

$$\begin{aligned}
& P_M - \mathbb{E}_0^l\left[\widetilde{\theta}\right] \\
= \; & \left(\pi_0^l + \pi_d^l - \pi_0^l\pi_d^l\right)\frac{\pi_0^h\pi_u^h + \left(\pi_0^h\left(1-\pi_u^h\right) + \left(1-\pi_0^h\right)\pi_d^h\right)\theta}{\pi_0^h + \pi_d^h - \pi_0^h\pi_d^h} - \pi_0^l\pi_u^l - \left(\pi_0^l\left(1-\pi_u^l\right) + \left(1-\pi_0^l\right)\pi_d^l\right)\theta \\
= \; & \left[\frac{\left(\pi_0^l + \pi_d^l - \pi_0^l\pi_d^l\right)\pi_0^h\pi_u^h}{\pi_0^h + \pi_d^h - \pi_0^h\pi_d^h} - \pi_0^l\pi_u^l\right](1-\theta).
\end{aligned}$$

Then, it is direct to use the market clearing condition to derive the equilibrium price listed in the proposition.

To show that the constraint (2) never binds in equilibrium, we only need to verify the case in which the optimists' equilibrium debt promise $F^* > \theta^2$. This occurs only when $p_0 \leq P_M$. From (30), an optimist's marginal value of using \$1 to purchase the risky asset (by borrowing with a optimal promise of $F^* = \theta$) is

$$\begin{aligned}
v_0^* &= \frac{\left[\pi_0^h\pi_u^h\left(1-F^*\right) + \left(\pi_0^h\left(1-\pi_u^h\right) + \left(1-\pi_0^h\right)\pi_d^h\right)\left(\theta - F^*\right)\right]}{p_0 - \theta^2 - \left(\pi_0^l + \pi_d^l - \pi_0^l\pi_d^l\right)\left(F^* - \theta^2\right)} \\
&\geq \frac{\pi_0^h\pi_u^h\left(1-\theta\right)}{P_M - \theta^2 - \left(\pi_0^l + \pi_d^l - \pi_0^l\pi_d^l\right)\left(\theta - \theta^2\right)} \\
&= \frac{\pi_0^h\pi_u^h\left(1-\theta\right)}{\left(\pi_0^l + \pi_d^l - \pi_0^l\pi_d^l\right)\frac{\pi_0^h\pi_u^h + \left(\pi_0^h\left(1-\pi_u^h\right) + \left(1-\pi_0^h\right)\pi_d^h\right)\theta}{\pi_0^h + \pi_d^h - \pi_0^h\pi_d^h} - \left(\pi_0^l + \pi_d^l - \pi_0^l\pi_d^l\right)\theta} \\
&= \frac{\pi_0^h + \pi_d^h - \pi_0^h\pi_d^h}{\pi_0^l + \pi_d^l - \pi_0^l\pi_d^l}.
\end{aligned}$$

On the other hand, if the optimist wants to acquire the risky debt with payoff $\widetilde{\theta} \wedge F^*$, one can directly verify that it is optimal to finance the position by using riskless debt with a

promise of θ^2. Then, his expected value is

$$v_0' = \frac{\left(\pi_0^h + \pi_d^h - \pi_0^h \pi_d^h\right)\left(F^* - \theta^2\right)}{\mathbb{E}_0^l\left[\widetilde{\theta} \wedge F^*\right] - \theta^2} = \frac{\pi_0^h + \pi_d^h - \pi_0^h \pi_d^h}{\pi_0^l + \pi_d^l - \pi_0^l \pi_d^l} \tag{32}$$

which is demonstrated by v_0^*. Therefore, the constraint in (2) always holds.

4.3 Proof of Proposition 3

Suppose that in equilibrium some optimists use a risky debt contract with a $F^* \in \left(\theta^2, \theta\right]$ from pessimists. Then, their marginal value from \$1 is

$$v_0^* = \frac{\mathbb{E}_0^h\left[\widetilde{\theta} - \widetilde{\theta} \wedge F^*\right]}{p_0 - \mathbb{E}_0^l\left[\widetilde{\theta} \wedge F^*\right]} = \frac{\left[\pi_0^h \pi_u^h \left(1 - F^*\right) + \left(\pi_0^h\left(1 - \pi_u^h\right) + \left(1 - \pi_0^h\right)\pi_d^h\right)\left(\theta - F^*\right)\right]}{p_0 - \mathbb{E}_0^l\left[\widetilde{\theta} \wedge F^*\right]}. \tag{33}$$

Because $p_0 \geq \mathbb{E}_0^l\left[\widetilde{\theta}\right]$, v_0^* reaches its maximum when

$$p_0 = \mathbb{E}_0^l\left[\widetilde{\theta}\right] \text{ and } F^* = \theta.$$

Thus, we have

$$\begin{aligned} v_0^* &\leq \frac{\pi_0^h \pi_u^h \left(1 - \theta\right)}{\mathbb{E}_0^l\left[\widetilde{\theta}\right] - \theta^2 - \left(\pi_0^l + \pi_d^l - \pi_0^l \pi_d^l\right)\left(\theta - \theta^2\right)} \\ &= \frac{\pi_0^h \pi_u^h \left(1 - \theta\right)}{\pi_0^l \pi_u^l \left(1 - \theta^2\right) + \left(\pi_0^l\left(1 - \pi_u^l\right) + \left(1 - \pi_0^l\right)\pi_d^l\right)\left(\theta - \theta^2\right) - \left(\pi_0^l + \pi_d^l - \pi_0^l \pi_d^l\right)\left(\theta - \theta^2\right)} \\ &= \frac{\pi_0^h \pi_u^h \left(1 - \theta\right)}{\pi_0^l \pi_u^l \left(1 - \theta^2\right) - \pi_0^l \pi_u^l \left(\theta - \theta^2\right)} = \frac{\pi_0^h \pi_u^h}{\pi_0^l \pi_u^l} \end{aligned}$$

Then, it is direct to see that v_0' in (32) dominates v_0^* in (33). Therefore, the constraint in (2) is always violated if there is any risky debt in equilibrium. In the absence of risky debt, it is direct to derive the market equilibrium described in the proposition.

4.4 Proof of Proposition 4

Suppose that the creditor has an equivalent belief of ϕ_d, which will be determined later in equilibrium and can be π_d^l if he is a pessimist. If the optimist promises to pay F on date 2, he can obtain the following credit:

$$D_d^S\left(F\right) = \begin{cases} F & \text{if } F \leq \theta^2 \\ \phi_d F + \left(1 - \phi_d\right)\theta^2 & \text{if } F \in \left[\theta^2, \theta\right] \end{cases}.$$

This debt contract allows him to use \$1 to establish a position of $1/\left(p_d - D_d^S(F)\right)$ units of the risky asset with an expected profit of

$$V(F) = \frac{\pi_d^h(\theta - F \wedge \theta) + \left(1 - \pi_d^h\right)\left(\theta^2 - F \wedge \theta^2\right)}{p_d - \phi_d(F \wedge \theta) - (1 - \phi_d)\left(F \wedge \theta^2\right)}.$$

If $F \leq \theta^2$,

$$V(F) = \frac{\pi_d^h \theta + \left(1 - \pi_d^h\right)\theta^2 - F}{p_d - F}$$

which is clearly increasing in F as $p_d \leq \pi_d^h \theta + \left(1 - \pi_d^h\right)\theta^2$. This is because a higher riskless promise allows the optimist to establish a greater position without incurring a higher financing cost.

If $F > \theta^2$,

$$V(F) = \frac{\pi_d^h(\theta - F)}{p_d - \theta^2 - \phi_d\left(F - \theta^2\right)}.$$

It is direct to show that

$$\frac{dV(F)}{dF} \propto -\pi_d^h\left[p_d - \left(\phi_d \theta + (1 - \phi_d)\theta^2\right)\right].$$

As the asset price p_d has to be above the asset valuation of the creditor ($p_d \geq \phi_d \theta + (1 - \phi_d)\theta^2$), we have $\frac{dV(F)}{dF} \leq 0$. In fact, if $p_d = \phi_d \theta + (1 - \phi_d)\theta^2$ (i.e., the creditor is also the marginal investor of the asset), the optimist is indifferent between any promise F in $\left[\theta^2, \theta\right]$. This is because the risky part of the debt has the same valuation as the asset, which makes the optimist indifferent between borrowing risky debt and taking a greater position in the asset.

Overall, the optimist's expected profit is maximized at $F = \theta^2$.

4.5 Proof of Proposition 5

Equations (11) and (12) are simply reformulations of the definitions of v_u and v_d in (7) and (6).

Equations (8) and (9) imply that

$$\begin{aligned} p_0 &= \frac{\pi_0^h v_u}{u_0} p_u + \left(1 - \frac{\pi_0^h v_u}{u_0}\right) p_d \\ &\geq \frac{\pi_0^h v_u}{v_0} p_u + \left(1 - \frac{\pi_0^h v_u}{v_0}\right) p_d \end{aligned} \tag{34}$$

i.e., p_0 has to be above the optimists' asset valuation derived from their marginal value of cash in states u and d of date 1. On the other hand, the pessimists' deep pockets imply their asset valuation as another lower bound for p_0:

$$p_0 \geq \pi_0^l p_u + \left(1 - \pi_0^l\right) p_d.$$

Combining these two inequalities leads to

$$p_0 \geq \phi_0 p_u + (1 - \phi_0) p_d,$$

with $\phi_0 = \max\left(\frac{\pi_0^h v_u}{v_0}, \pi_0^l\right)$. Furthermore, the equality must bind. Suppose that $p_0 > \phi_0 p_u + (1 - \phi_0) p_d$. Then, it must be that (34) holds strictly, i.e., $v_0 = \pi_0^h v_u + \left(1 - \pi_0^h\right) v_d > u_0$, which in turn implies that optimists prefer saving cash on date 0. Then, the marginal investor of the asset has to be pessimist. However, $p_0 > \phi_0 p_u + (1 - \phi_0) p_d \geq \pi_0^l p_u + \left(1 - \pi_0^l\right) p_d$, which is a contradiction. Thus, (13) has to hold.

4.6 Proof of Proposition 6

Follow up with the example considered in the main text regarding the debt contract in state d with a promise to pay F on date 2. Suppose that $D_d^S(F) < \phi_d (F \wedge \theta) + (1 - \phi_d)\left(F \wedge \theta^2\right)$. If $F < \theta^2$, then the debt is riskless, and the debt price constitutes arbitrage to the marginal investor of the asset. Suppose that $F \in \left(\theta^2, \theta\right)$, i.e., the debt is risky. If the marginal investor of the asset is pessimist (i.e., $\phi_d = \pi_d^l$), then any pessimist will be able to bid up the debt price. Thus, this cannot hold in equilibrium. If the marginal investor is optimists, by borrowing $F \wedge \theta^2$ to purchase the debt contract, an optimist can reach a value that is strictly above the marginal value of cash v_d in state d:

$$\frac{\pi_d^h\left(F \wedge \theta - F \wedge \theta^2\right)}{D_d^S(F) - F \wedge \theta^2} > \frac{\pi_d^h}{\phi_d} = v_d.$$

This is again a contradiction. Thus, $D_d^S(F) < \phi_d (F \wedge \theta) + (1 - \phi_d)\left(F \wedge \theta^2\right)$ cannot occur.

Neither can $D_d^S(F) > \phi_d (F \wedge \theta) + (1 - \phi_d)\left(F \wedge \theta^2\right)$ occur, because this valuation is too high for anyone to pay.

4.7 Proof of Proposition 7

It is clear that for either long-term or short-term debt contracts, the credit the optimist can obtain on date 0 monotonically increases with the promise. As the promise, increases from

0 to 1, the credit increases from 0 to $\phi_0 p_u + (1-\phi_0) p_d$, which is the market valuation of the collateral. Thus, the optimist can find the necessary promise to obtain any credit level in this range by using either a long-term contract or a short-term contract.

We now prove that the market value of the long-term contract is more sensitive to the date-1 state than the short-term contract. Suppose the long-term debt gives a promise of F_0^L and short-term debt gives a promise of F_0^S. Then, equation (17) implies that

$$\phi_0 \left[\left(D_u^S - D_d^S \right) - \left(D_u^L - D_d^L \right) \right] = D_d^L - D_d^S.$$

Thus, we only need to show that $D_d^L - D_d^S \leq 0$. This inequality is equivalent to $D_d^S \left(F_0^L \right) - F_0^S \wedge p_d \leq 0$. Suppose that it does not hold. Then, $D_d^S \left(F_0^L \right) > F_0^S \wedge p_d$. Since $D_d^S \left(F_0^L \right) \leq p_d$, we must have $p_d > F_0^S$ and $D_d^S \left(F_0^L \right) > F_0^S$. Since $p_u > p_d > F_0^S$, $\phi_0 \left(F_0^S \wedge p_u \right) + (1-\phi_0) \left(F_0^S \wedge p_d \right) = F_0^S$, but the market value of long-term debt is $D_d^S \left(F_0^L \right) + \phi_0 (D_u^S \left(F_0^L \right) - D_d^S \left(F_0^L \right)) > F_0^S$. This contradicts the initial assumption that the long-term and short-term contracts have the same value.

4.8 Proof of Proposition 8

The optimist's expected value of saving one dollar is

$$\pi_0^h v_u + \left(1 - \pi_0^h \right) v_d.$$

If the optimist uses a long-term debt contract with a promise of F_L to establish an asset position, his expected value from applying (16) is

$$\begin{aligned} & \left[\pi_0^h v_u + \left(1 - \pi_0^h \right) v_d \right] \frac{\widehat{\pi}_0^h \left(p_u - D_u^L \right) + \left(1 - \widehat{\pi}_0^h \right) \left(p_d - D_d^L \right)}{p_0 - \phi_0 D_u^L - (1-\phi_0) D_d^L} \\ = \; & \left[\pi_0^h v_u + \left(1 - \pi_0^h \right) v_d \right] \frac{\widehat{\pi}_0^h \left(p_u - D_u^L \right) + \left(1 - \widehat{\pi}_0^h \right) \left(p_d - D_d^L \right)}{\phi_0 \left(p_u - D_u^L \right) + (1-\phi_0) \left(p_d - D_d^L \right)}, \end{aligned}$$

One can directly verify that

$$p_u - p_d \geq D_u^L - D_d^L,$$

i.e., the difference of the optimist's equity value across the up and down state is higher than that of the market value of the long-term debt contract. Then, we can show that

$$\begin{aligned} & \widehat{\pi}_0^h \left(p_u - D_u^L \right) + \left(1 - \widehat{\pi}_0^h \right) \left(p_d - D_d^L \right) - \left[\phi_0 \left(p_u - D_u^L \right) + (1-\phi_0) \left(p_d - D_d^L \right) \right] \\ = \; & \left(\widehat{\pi}_0^h - \phi_0 \right) \left(p_u - p_d - \left(D_u^L - D_d^L \right) \right) \leq 0, \end{aligned}$$

as $\widehat{\pi}_0^h < \phi_0$. Therefore,

$$\left[\pi_0^h v_u + \left(1-\pi_0^h\right) v_d\right] \frac{\widehat{\pi}_0^h \left(p_u - D_u^L\right) + \left(1-\widehat{\pi}_0^h\right)\left(p_d - D_d^L\right)}{\phi_0 \left(p_u - D_u^L\right) + \left(1-\phi_0\right)\left(p_d - D_d^L\right)} < \pi_0^h v_u + \left(1-\pi_0^h\right) v_d.$$

Thus, saving capital is dominant.

4.9 Proof of Proposition 9

Suppose that $\widehat{\pi}_0^h > \phi_0$. Since $p_0 = \phi_0 p_u + (1-\phi_0) p_d$, we have

$$p_0 < \widehat{\pi}_0^h p_u + \left(1-\widehat{\pi}_0^h\right) p_d.$$

Then the optimist's expected value from using short-term debt to establish a position, e.g. (16), is

$$\left[\pi_0^h v_u + \left(1-\pi_0^h\right) v_d\right] \left[\frac{\widehat{\pi}_0^h p_u + \left(1-\widehat{\pi}_0^h\right) p_d - \widehat{\pi}_0^h \left(F_0^S \wedge p_u\right) - \left(1-\widehat{\pi}_0^h\right)\left(F_0^S \wedge p_d\right)}{p_0 - \phi_0 \left(F_0^S \wedge p_u\right) - \left(1-\phi_0\right)\left(F_0^S \wedge p_d\right)}\right]. \quad (35)$$

As in the proof of Proposition 4, we can easily show that it is optimal to use $F_0^S = p_d$, and the second bracket becomes

$$\frac{\widehat{\pi}_0^h \left(p_u - p_d\right)}{p_0 - p_d} > 1.$$

As a result, by using a short-term debt contract with a promise of p_d to purchase the asset, the optimist obtains a value strictly higher than that from saving cash.

The equivalence of cash saving and establishing position is trivial when $\widehat{\pi}_0^h = \phi_0$, as (35) collapses to $\pi_0^h v_u + \left(1-\pi_0^h\right) v_d$, which is the value from saving cash.

4.10 Proof of Lemma 10

Suppose that every optimist takes a position on date 0 by using one-period debt contract with a promise of p_d. Then, all optimists will be wiped out in state d of date 1, and thus $p_d = \mathbb{E}_d^l\left[\widetilde{\theta}\right]$. This price in turn implies that optimists' marginal value of cash in the state reaches its maximum: $v_d = \pi_d^h / \pi_d^l$. Then, an optimist's cash-value-adjusted belief on date 0 is

$$\widehat{\pi}_0^h = \frac{\pi_0^h}{\pi_0^h + \left(1-\pi_0^h\right) v_d} = \frac{\pi_0^h}{\pi_0^h + \left(1-\pi_0^h\right) \pi_d^h / \pi_d^l}. \quad (36)$$

By the inequality in (28), we can show that

$$\frac{\pi_0^h + \left(1-\pi_0^h\right) \pi_d^h}{\pi_0^l + \left(1-\pi_0^l\right) \pi_d^l} < \pi_0^h + \left(1-\pi_0^h\right) \pi_d^h / \pi_d^l,$$

which is equivalent to

$$\pi_0^h \left[\frac{1}{\pi_0^l + \left(1 - \pi_0^l\right) \pi_d^l} - 1 \right] < \left(1 - \pi_0^h\right) \pi_d^h \left[\frac{1}{\pi_d^l} - \frac{1}{\pi_0^l + \left(1 - \pi_0^l\right) \pi_d^l} \right],$$

which in turn is equivalent to a direct implication of the inequality in (28):

$$\frac{\pi_0^h}{\pi_0^l} < \frac{\left(1 - \pi_0^h\right) \pi_d^h}{\left(1 - \pi_0^l\right) \pi_d^l}.$$

Then, equation (36) implies that $\widehat{\pi}_0^h < \pi_0^l \leq \phi_0$, which in turn motivates the optimist to save cash according to Proposition 8. This is a contradiction. Thus, at least some optimists have to save cash on date 0 so that the asset's price in state d won't fall too low.

4.11 Proof of Proposition 11

We focus on cases 2 and 3.

Case 2 In this case, $\phi_d \geq \pi_d^l$ and $\phi_0 \geq \pi_0^l$. Thus, $\lambda = 1$, and equations (25) and (27) imply that

$$\left.\begin{array}{c} p_d = (1 - \alpha) c + \theta^2 \\ p_0 = \alpha c + p_d \end{array}\right\} \Rightarrow p_0 = c + \theta^2.$$

To determine ϕ_d, note that

$$p_0 = \frac{\pi_0^h}{\pi_0^h + \left(1 - \pi_0^h\right) v_d} p_u + \frac{\left(1 - \pi_0^h\right) v_d}{\pi_0^h + \left(1 - \pi_0^h\right) v_d} p_d. \tag{37}$$

Proposition 5 implies $p_d = \theta^2 + \phi_d \left(\theta - \theta^2\right)$, and Proposition 4 implies that

$$v_d = \frac{\pi_d^h \left(\theta - \theta^2\right)}{p_d - \theta^2} = \frac{\pi_d^h}{\phi_d}.$$

By substituting these equations back to (37), we obtain

$$p_0 \left[\pi_0^h \phi_d + \left(1 - \pi_0^h\right) \pi_d^h\right] = \pi_0^h p_u \phi_d + \left(1 - \pi_0^h\right) \pi_d^h \left[\theta^2 + \phi_d \left(\theta - \theta^2\right)\right],$$

which implies that

$$\begin{aligned} \phi_d &= \frac{\left(1 - \pi_0^h\right) \pi_d^h \left(p_0 - \theta^2\right)}{\pi_0^h \left(p_u - p_0\right) + \left(1 - \pi_0^h\right) \pi_d^h \left(\theta - \theta^2\right)} \\ &= \frac{c \left(1 - \pi_0^h\right) \pi_d^h}{\pi_0^h \left(p_u - c - \theta^2\right) + \left(1 - \pi_0^h\right) \pi_d^h \left(\theta - \theta^2\right)}. \end{aligned} \tag{38}$$

Therefore,

$$\begin{aligned} p_d &= \theta^2 + \phi_d\left(\theta - \theta^2\right) = \theta^2 + \frac{c\left(\theta - \theta^2\right)\left(1 - \pi_0^h\right)\pi_d^h}{\pi_0^h\left(p_u - c - \theta^2\right) + \left(1 - \pi_0^h\right)\pi_d^h\left(\theta - \theta^2\right)}, \\ \alpha &= \frac{p_0 - p_d}{c}. \end{aligned}$$

Then, in order for $\phi_0 \geq \pi_0^l$, we require

$$\phi_0 = \frac{\pi_0^h}{\pi_0^h + \left(1 - \pi_0^h\right)\pi_d^h/\phi_d} \geq \pi_0^l,$$

which is equivalent to

$$\phi_d \geq \phi_d^* \equiv \frac{\pi_0^l\left(1 - \pi_0^h\right)\pi_d^h}{\left(1 - \pi_0^l\right)\pi_0^h}. \tag{39}$$

Note that $\phi_d^* \geq \pi_d^l$ because of the inequality condition in (29). Thus, as c falls, ϕ_0 hits π_0^l before ϕ_d hits π_d^l. By substituting (38) into (39), we obtain a lower bound for c to ensure that $\phi_0 \geq \pi_0^l$:

$$c \geq \frac{\pi_0^l}{\pi_0^h}\left[\pi_0^h\left(p_u - \theta^2\right) + \left(1 - \pi_0^h\right)\pi_d^h\left(\theta - \theta^2\right)\right].$$

Case 3 In this case, c goes down further. In this case, ϕ_0 already reaches its lower bound π_0^l. This in turn implies that pessimists become indifferent between holding or selling their asset endowments. The lower c also makes ϕ_d lower and thus marginal value of cash $v_d = \pi_d^h/\phi_d$ higher, which motivates optimists to save cash. Thus, in order for optimists to purchase the asset sold by pessimists, we need

$$\widehat{\pi}_0^h = \frac{\pi_0^h}{\pi_0^h + \left(1 - \pi_0^h\right)\pi_d^h/\phi_d} = \pi_0^l.$$

This implies that

$$\phi_d = \phi_d^*$$

where ϕ_d^* is defined in (39). Thus, both p_0 and p_d stay at constant levels as c falls down:

$$\begin{aligned} p_d &= p_d^* \equiv \theta^2 + \frac{\pi_0^l\left(1 - \pi_0^h\right)\pi_d^h}{\pi_0^h\left(1 - \pi_0^l\right)}\left(\theta - \theta^2\right) > \mathbb{E}_d^l\left[\widetilde{\theta}\right], \\ p_0 &= p_0^* \equiv \pi_0^l p_u + \left(1 - \pi_0^l\right)p_d^* > \mathbb{E}_0^l\left[\widetilde{\theta}\right]. \end{aligned}$$

As c falls, the adjustment in the equilibrium is through λ, the fraction of pessimists who sell their asset endowments. The market clearing condition in state d (equation (24)) implies that

$$p_d = (1 - \alpha)c + \theta^2 \Rightarrow \alpha = 1 - \frac{p_d^* - \theta^2}{c}.$$

Thus, α drops as c falls, i.e., more optimists choose to save cash on date 0. The market clearing condition in state d (equation (27)) implies that

$$\lambda p_0 = \alpha c + \lambda p_d \Rightarrow \lambda = \frac{\alpha c}{p_0^* - p_d^*}.$$

Thus, λ also drops as c falls, i.e., more pessimists choose to hold their asset endowments.

c needs to be above a lower bound in order for $\alpha \geq 0$:

$$c \geq p_d^* - \theta^2 = \frac{\pi_0^l \left(1 - \pi_0^h\right) \pi_d^h}{\pi_0^h \left(1 - \pi_0^l\right)} \left(\theta - \theta^2\right).$$

4.12 Proof of Proposition 12

The proof follows. Suppose that the equilibrium price of the asset exists and is denoted by $p_{n,j}$. As suggested by our analysis of the two-period setting, the asset price determines the optimists' investment opportunities and thus marginal value of cash. The asset price also offers a benchmark to set the market value of debt issued by the optimists. First, note that since the pessimists always have sufficient capital, the asset price on any state $p_{n,j}$ must be bounded from below by the pessimists' expected next-period price:

$$p_{n,j} \geq p_{n+1,j} + \pi_{n,j}^l \left(p_{n+1,j+1} - p_{n+1,j}\right), \tag{40}$$

with the equality binding when pessimists take asset positions and thus are the marginal investor of the asset.

The scarcity of optimist capital implies that the optimists can earn rent on their capital. Let's denote their marginal value of cash by $v_{n,j}$ in state j of date n. Suppose that an optimist has one dollar in this state. He can invest it in the asset and lever up the position using the maximum riskless one-period debt (with a promise of $p_{n+1,j}$), which gives him a position of $1/\left(p_{n,j} - p_{n+1,j}\right)$ shares and a per-share profit of $p_{n+1,j+1} - p_{n+1,j}$ in the up state of the next period and 0 in the down state. Given the optimist's belief $\pi_{n,j}^h$ about the up state and the marginal value of capital in the state as $v_{n+1,j+1}$, the expected value from having the position is

$$u_{n,j} = \frac{\pi_{n,j}^h v_{n+1,j+1} \left(p_{n+1,j+1} - p_{n+1,,j}\right)}{p_{n,j} - p_{n+1,j}}. \tag{41}$$

Given the price lower bound in (40),

$$u_{n,j} \leq \frac{\pi_{n,j}^h v_{n+1,j+1}}{\pi_{n,j}^l}.$$

In other words, the expected value from establishing the position is bounded by the pessimists' contemporaneous belief.

Alternatively, the optimist can also choose to save his cash for the next period, which gives him an expected value of $\pi_{n,j}^{h} v_{n+1,j+1} + \left(1 - \pi_{n,j}^{h}\right) v_{n+1,j}$. Thus, his marginal value of cash in the state is

$$v_{n,j} = \max\left(u_{n,j}, \quad \pi_{n,j}^{h} v_{n+1,j+1} + \left(1 - \pi_{n,j}^{h}\right) v_{n+1,j}\right).$$

This equation implies that the optimists' marginal value of cash is a supermartingale: $v_{n,j} \geq \mathbb{E}\left[v_{n+1,\cdot}\right]$, as they have the option to save, and that $v_{n,j} \geq u_{n,j}$. The latter, together with (41), implies that

$$p_{n,j} \geq p_{n+1,j} + \frac{\pi_{n,j}^{h} v_{n+1,j+1}}{v_{n,j}} \left(p_{n+1,j+1} - p_{n+1,j}\right), \tag{42}$$

with the equality binding when the optimists take positions in the asset and are the marginal investors. One of the equalities in (40) and (42) should bind in any given state. Thus, we have

$$p_{n,j} = p_{n+1,j} + \phi_{n,j} \left(p_{n+1,j+1} - p_{n+1,j}\right), \tag{43}$$

where

$$\phi_{n,j} = \max\left\{\frac{\pi_{n,j}^{h} v_{n+1,j+1}}{v_{n,j}}, \pi_{n,j}^{l}\right\}. \tag{44}$$

Note that $\phi_{n,j} \in (0,1)$. This process $\left\{\phi_{n,j}\right\}$, which accounts for the alternation of the marginal investor between the optimists and pessimists, acts as a risk-neutral probability for representing the equilibrium asset price.

This representation of the equilibrium asset price also applies to any contingent claim whose price is fully determined by the state of the binomial tree, such as any debt contract collateralized by the asset. Denote its market price in state j of date n by $q_{n,j}$. Then, this price has to satisfy

$$q_{n,j} = q_{n+1,j} + \phi_{n,j} \left(q_{n+1,j+1} - q_{n+1,j}\right). \tag{45}$$

To verify this, suppose that $q_{n,j} < q_{n+1,j} + \phi_{n,j} \left(q_{n+1,j+1} - q_{n+1,j}\right)$. Then, the marginal investor of the asset, either an optimist (if $\frac{\pi_{n,j}^{h} v_{n+1,j+1}}{v_{n,j}} > \pi_{n,j}^{l}$) or a pessimist (if $\frac{\pi_{n,j}^{h} v_{n+1,j+1}}{v_{n,j}} < \pi_{n,j}^{l}$) or both (if $\frac{\pi_{n,j}^{h} v_{n+1,j+1}}{v_{n,j}} = \pi_{n,j}^{l}$), would find buying this debt contract more attractive than buying the asset. This is a contradiction. Now suppose that $q_{n,j} > q_{n+1,j} + \phi_{n,j} \left(q_{n+1,j+1} - q_{n+1,j}\right)$. The expected return across the following period implied by this price is lower than the asset

to any market participant. As a result, no one would be willing to buy this debt contract. Again, this is a contradiction. Thus, (45) has to hold.

Based on the representation in (43) and (45), we provide a lemma to show that monotonicity of asset prices is preserved across different dates on the tree.

Lemma 13 *Consider any contingent claim on the asset. If its final payoff monotonically increases with respect to the state of the final date, then its price on any prior date also monotonically increases with the state.*

Proof. *Without loss of generality, we consider two states j and $j+1$ on date n, which is prior to its maturity date. Then, the price of the claim in these states are*

$$\begin{aligned} q_{n,j} &= q_{n+1,j} + \phi_{n,j}\left(q_{n+1,j+1} - q_{n+1,j}\right), \\ q_{n,j+1} &= q_{n+1,j+1} + \phi_{n,j+1}\left(q_{n+1,j+2} - q_{n+1,j+1}\right). \end{aligned}$$

Then,

$$q_{n,j+1} - q_{n,j} = \left(1 - \phi_{n,j}\right)\left(q_{n+1,j+1} - q_{n+1,j}\right) + \phi_{n,j+1}\left(q_{n+1,j+2} - q_{n+1,j+1}\right).$$

Thus, if $q_{n+1,j+2} \geq q_{n+1,j+1} \geq q_{n+1,j}$, then $q_{n,j+1} \geq q_{n,j}$. Using this property, backward induction on the tree implies that the monotonicity is preserved across any date. ■

This lemma directly implies the price of the asset $p_{n,j}$ and the market value of any long-term debt contract $D_{n,j}$ collateralized by one share of the asset monotonically increase with the state on the tree. Furthermore, note that on the maturity date of the debt contract, the difference before the asset price and the debt payoff also monotonically increases with the state. Thus, this lemma implies that across any two adjacent states j and $j+1$ on any date n prior to the debt maturity, $p_{n,j+1} - D_{n,j+1} \geq p_{n,j} - D_{n,j}$, which, in turn, implies that

$$p_{n,j+1} - p_{n,j} \geq D_{n,j+1} - D_{n,j}. \tag{46}$$

This inequality suggests that the asset price is more sensitive to the state than price of any debt contract collateralized by the asset.

We are ready to show that in any state on the tree, an individual optimist finds it optimal to either acquire an asset position by using the maximum riskless one-period debt or to simply save cash for a future period. Consider state j of date n. Define the optimist's cash-value-adjusted belief by

$$\widehat{\pi}^h_{n,j} \equiv \frac{\pi^h_{n,j} v_{n+1,j+1}}{\pi^h_{n,j} v_{n+1,j+1} + \left(1 - \pi^h_{n,j}\right) v_{n+1,j}}.$$

Suppose that $\widehat{\pi}^h_{n,j} \geq \pi^l_{n,j}$. Then, optimists are the marginal investors of the asset. We can verify that for an optimist, financing an asset position by using the maximum riskless short-term debt dominates over using any long-term debt contract. Let's compare the financing of a one-period debt contract with a promise of F^S on date $n+1$ and another long-term debt contract with a promise of F^L on date $N > n+1$. Denote the price of the short-term contract by $D^S_{n,j}$ and the price of the long-term contract by $D^L_{n,j}$. As both contracts, if issued, have to be purchased by the pessimists, to generate the same credit requires

$$\phi_{n,j} D^S_{n+1,j+1} + \left(1 - \phi_{n,j}\right) D^S_{n+1,j} = \phi_{n,j} D^L_{n+1,j+1} + \left(1 - \phi_{n,j}\right) D^L_{n+1,j}. \tag{47}$$

The short-term debt being less costly to the optimist is equivalent to

$$\widehat{\pi}^h_{n,j} D^S_{n+1,j+1} + \left(1 - \widehat{\pi}^h_{n,j}\right) D^S_{n+1,j} \leq \widehat{\pi}^h_{n,j} D^L_{n+1,j+1} + \left(1 - \widehat{\pi}^h_{n,j}\right) D^L_{n+1,j}. \tag{48}$$

By taking the difference between (48) and (47), (48) is equivalent to

$$\left(\widehat{\pi}^h_{n,j} - \phi_{n,j}\right) \left[\left(D^L_{n+1,j+1} - D^L_{n+1,j}\right) - \left(D^S_{n+1,j+1} - D^S_{n+1,j}\right)\right] \geq 0.$$

Because $\widehat{\pi}^h_{n,j} \geq \pi^l_{n,j}$, this holds if and only if

$$\left(D^L_{n+1,j+1} - D^L_{n+1,j}\right) - \left(D^S_{n+1,j+1} - D^S_{n+1,j}\right) \geq 0. \tag{49}$$

Equation (47) implies that

$$\left(D^L_{n+1,j+1} - D^L_{n+1,j}\right) - \left(D^S_{n+1,j+1} - D^S_{n+1,j}\right) = \frac{1}{\phi_{n,j}} \left(D^S_{n+1,j} - D^L_{n+1,j}\right).$$

Note that the short-term debt contract matures on $n+1$ and its payoffs are

$$D^S_{n+1,j+1} = F^S, \text{ and } D^S_{n+1,j} = \min\left(F^S, p_{n+1,j}\right).$$

Thus, (49) is equivalent to

$$\min\left(F^S, p_{n+1,j}\right) \geq D^L_{n+1,j}.$$

Suppose that $D^L_{n+1,j} > \min\left(F^S, p_{n+1,j}\right)$. As $D^L_{n+1,j} \leq p_{n+1,j}$, we must have $F^S < p_{n+1,j}$ and $F^S < D^L_{n+1,j}$. Then,

$$\begin{aligned}
& \phi_{n,j} D^S_{n+1,j+1} + \left(1 - \phi_{n,j}\right) D^S_{n+1,j} \\
= \; & \phi_{n,j} F^S + \left(1 - \phi_{n,j}\right) \min\left(F^S, p_{n+1,j}\right) \\
< \; & F^S < D^L_{n+1,j} \leq \phi_{n,j} D^L_{n+1,j+1} + \left(1 - \phi_{n,j}\right) D^L_{n+1,j}
\end{aligned}$$

which contradicts (47). Thus, (49) holds and the short-term contract dominates the long-term contract.

Now suppose that $\widehat{\pi}^h_{n,j} < \phi_{n,j}$. We can show that saving cash dominates over using any long-term debt to finance an asset position. Saving cash gives an expected value of

$$\pi^h_{n,j} v_{n+1,j+1} + \left(1 - \pi^h_{n,j}\right) v_{n+1,j}.$$

If an optimist uses a long-term contract to finance an asset position, the expected value is

$$\begin{aligned}
& \frac{\pi^h_{n,j} v_{n+1,j+1}\left(p_{n+1,j+1} - D^L_{n+1,j+1}\right) + \left(1 - \pi^h_{n,j}\right) v_{n+1,j}\left(p_{n+1,j} - D^L_{n+1,j}\right)}{p_{n,j} - \phi_{n,j} D^L_{n+1,j+1} - \left(1 - \phi_{n,j}\right) D^L_{n+1,j}} \\
= \; & \frac{\pi^h_{n,j} v_{n+1,j+1} + \left(1 - \pi^h_{n,j}\right) v_{n+1,j}}{p_{n,j} - \phi_{n,j} D^L_{n+1,j+1} - \left(1 - \phi_{n,j}\right) D^L_{n+1,j}} \left[\widehat{\pi}^h_{n,j}\left(p_{n+1,j+1} - D^L_{n+1,j+1}\right) + \left(1 - \widehat{\pi}^h_{n,j}\right)\left(p_{n+1,j} - D^L_{n+1,j}\right)\right] \\
= \; & \left[\pi^h_{n,j} v_{n+1,j+1} + \left(1 - \pi^h_{n,j}\right) v_{n+1,j}\right] \frac{\widehat{\pi}^h_{n,j}\left(p_{n+1,j+1} - D^L_{n+1,j+1}\right) + \left(1 - \widehat{\pi}^h_{n,j}\right)\left(p_{n+1,j} - D^L_{n+1,j}\right)}{\phi_{n,j}\left(p_{n+1,j+1} - D^L_{n+1,j+1}\right) + \left(1 - \phi_{n,j}\right)\left(p_{n+1,j} - D^L_{n+1,j}\right)} \\
\leq \; & \pi^h_{n,j} v_{n+1,j+1} + \left(1 - \pi^h_{n,j}\right) v_{n+1,j}.
\end{aligned}$$

The last inequality is because

$$\left(\phi_{n,j} - \widehat{\pi}^h_{n,j}\right)\left[\left(p_{n+1,j+1} - p_{n+1,j}\right) - \left(D^L_{n+1,j+1} - D^L_{n+1,j}\right)\right] \geq 0$$

because of (46). Thus, saving cash dominates.