Optimal Consumption and Investment Decisions with Disastrous Income Risk: Revisiting Rietz's Rare Disaster Risk Hypothesis

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Motivation

- How should disastrous income risk affect the optimal consumption and investment decisions of individuals?
- Precautionary savings consistent with the permanent income hypothesis

Motivation (Cont'd)

In particular, by specifying Mehra and Prescott's model to include ***a low-probability, depression-like third state****, I can explain both high equity risk premia and low risk-free returns without abandoning the Arrow-Debreu paradigm (Rietz, 1988)*

- We consider a version of the Merton (1969, 1971) model with the special feature that income can abruptly jump from a positive value to a smaller positive value or even to zero

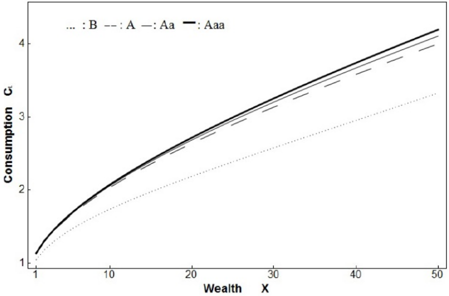
Motivation (Cont'd)

- The currently available social securities and private insurance market are insufficient to perfectly hedge against disastrous income risk (Cocco *et al.*, 2005; Bensoussan *et al.*, 2016; Jang *et al.*, 2019; Jang *et al.*, 2020)
- If there is an insurance market for (partially) hedging against disastrous income risk, the individual's income is partly wiped out when a disastrous income shock occurs

Direction

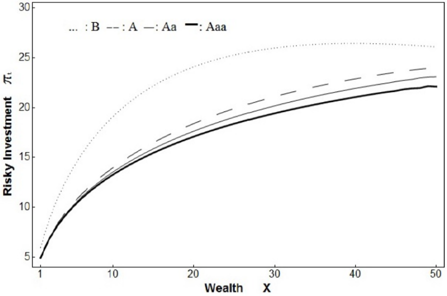
- We shed new light on dynamic models of optimal consumption and investment decisions for individuals who exhibit constant absolute risk aversion (CARA) utility preferences by exploring insights into how possibility of **a disastrous income shock** combined with a non-negative constraint on borrowing affects both the consumption/savings and wealth allocation decisions between bonds and equity

Main Findings



- A large precautionary savings motive
- A significant discontinuity and the dramatic change in the concavity of consumption

Main Findings (Cont'd)



- The precautionary savings terms' role in the risky investment
- Risky assets as a partial hedging tool against disastrous income risk in view of agents' liquidity

Main Findings (Cont'd)

$x \setminus \delta$	$k=0$				$k=0.1$				$k=0.25$			
	Aaa	Aa	A	B	Aaa	Aa	A	B	Aaa	Aa	A	B
1	1.1371	1.1300	1.1260	1.0469	1.1326	1.1311	1.1288	1.0930	1.1327	1.1323	1.1322	1.1664
10	2.0755	2.0596	2.0366	1.7398	2.0748	2.0636	2.0461	1.8023	2.0752	2.0681	2.0576	1.8894
20	2.7194	2.6871	2.6410	2.1895	2.7195	2.6945	2.6575	2.2612	2.7203	2.7030	2.6778	2.3641
30	3.2571	3.2064	3.1379	2.5806	3.2578	3.2173	3.1606	2.6578	3.2590	3.2301	3.1892	2.7841
40	3.7427	3.6732	3.5845	2.9552	3.7439	3.6875	3.6125	3.0359	3.7456	3.7044	3.6484	3.1890
50	4.1971	4.1093	4.0034	3.3295	4.1988	4.1268	4.0357	3.4116	4.2009	4.1477	4.0777	3.5735

(A) consumption

$x \setminus \delta$	$k=0$				$k=0.1$				$k=0.25$			
	Aaa	Aa	A	B	Aaa	Aa	A	B	Aaa	Aa	A	B
1	4.9360	4.9024	4.9514	5.9681	4.8716	4.8936	4.9301	5.9793	4.8701	4.8843	4.9078	6.2262
10	13.2873	13.5563	13.9905	19.1176	13.2675	13.4948	13.8481	18.6576	13.2612	13.4244	13.6769	18.0992
20	17.0904	17.6332	18.3955	24.0844	17.0719	17.5228	18.1680	23.6951	17.0598	17.3932	17.8819	22.6893
30	19.4330	20.1824	21.1097	25.9751	19.4125	20.0400	20.8520	25.6721	19.3955	19.8677	20.5128	23.9466
40	21.0595	21.9355	22.8843	26.4341	21.0367	21.7729	22.6365	26.2535	21.0165	21.5840	22.2962	24.7958
50	22.1026	23.0891	24.0704	26.0817	22.0784	22.9668	23.8046	26.0728	22.0596	22.8175	23.5572	26.6611

(B) risky investment

- The role of of income recovery after the occurrence of income disaster in optimal decisions

Risk Management

- Significance of the low-probability, high-impact aspect of disastrous income risk
- Large and negative earnings losses are observed at job displacement (Low *et al.*, 2010)
- Such substantial losses have a large impact on household investment and consumption decisions (Guvenen *et al.* 2015)
- Focus on the extremes of the probability distribution of income, deviating from log-normality substantially

Rare Disaster Risk Hypothesis

- Extending the seminal study of Rietz (1988), Barro (2006), Gabaix (2008, 2012), Wachter (2013), Pindyck and Wang (2013), Farhi and Gabaix (2016), Barro *et al.* (2022), Hong *et al.* (2023) develop different rare disaster models having focuses on asset pricing implications based on general equilibrium models
- The complete-markets general equilibrium economy v.s. the incomplete-markets partial equilibrium environment
- Empirical regularities (e.g., the equity premium puzzle, the risk-free rate puzzle) with general equilibrium models v.s. optimal consumption/savings and investment behaviors with disastrous income risk with a partial equilibrium model

Related Literature

- Cocco *et al.* (2005): the role of market incompleteness caused by uninsurable labor income risk in individuals' optimal policies
- Bensoussan *et al.* (2016): the effects of the risk of forced unemployment on interdependent consumption/savings, portfolio selection retirement decisions
- Wang *et al.* (2016): the impact of stochastic income on optimal consumption and savings decisions with recursive utility
- Our paper: the relations among state-dependent and stochastically time-varying income disasters, consumption/savings, and portfolio choice

Model Settings

- The CARA utility preference

$$U = E\Big[\int_0^\infty e^{-\beta t}\Big(-\frac{1}{\gamma}e^{-\gamma c_t}\Big)dt\Big]$$

- A riskless bond and a risky stock

$$dB_t = rB_t dt,$$
$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

- The deterministic labor income stream

$$d\epsilon_t = \mu^\epsilon \epsilon_t dt, \quad \epsilon_0 = \epsilon > 0$$

Model Settings (Cont'd)

- The value function

$$V(x,\epsilon) \equiv \max_{(c,\pi)} E\Big[\int_0^\infty e^{-\beta t}\Big(-\frac{1}{\gamma}e^{-\gamma c_t}\Big)dt\Big],$$

subject to

$$dX_t = (rX_t - c_t + \epsilon_t)dt + \pi_t\sigma(dW_t + \theta dt),\ \ \theta = \frac{\mu - r}{\sigma},\ \ X_0 = x > -\frac{\epsilon}{r^\epsilon},$$

where

$$r^\epsilon = r - \mu^\epsilon$$

Hamilton-Jacobi-Bellman (HJB) Equation

- The Hamilton-Jacobi-Bellman (HJB) equation

$$\max_{(c,\pi)}\Big[-\beta V(x,\epsilon)+(rx-c+\epsilon)V_x(x,\epsilon)+\frac{1}{2}\pi^2\sigma^2V_{xx}(x,\epsilon)$$
$$+\pi\sigma\theta V_x(x,\epsilon)+\mu^\epsilon\epsilon V_\epsilon(x,\epsilon)-\frac{1}{\gamma}e^{-\gamma c}\Big]=0$$

- Solution

$$V(x,\epsilon)=-\frac{A}{\gamma r}e^{-\gamma r(x+a\epsilon)},$$

where

$$A=e^{-\frac{1}{r}\left(\frac{\theta^2}{2}+\beta-r\right)},\quad a=\frac{1}{r^\epsilon}$$

Optimal Strategies

- The optimal consumption and investment strategies

$$c = r\left[x + \frac{\epsilon}{r^{\epsilon}} + \frac{\theta^2}{2\gamma r^2}\left(1 + \frac{2}{\theta^2}(\beta - r)\right)\right]$$

$$\pi = \frac{\theta}{\gamma\sigma}\frac{1}{r}$$

- The affine structure of the optimal consumption in total wealth
- The *wealth effect* issue in the optimal investment

Three General Models

- *Mode 1*: The basic model with borrowing constraints
- *Model 2*: Model 1 with a one-time-only disastrous income shock
- *Model 3*: Model 2 with state-dependent and time-varying disastrous income risk

Model 1

- Borrowing constraints due to market frictions (e.g., informational asymmetry, agency conflicts, limited enforcement)
- In the presence of borrowing constraints,

$$X_t \geq 0 \quad \text{for all} \quad t \geq 0$$

- In the presence of borrowing constraints, the HJB equation is no longer separable in wealth x and income ϵ due to the wealth effect

Model 1: Convex-Dual Approach

- A modified convex-duality approach of Bensoussan *et al.* (2016)
- The dual variable and the convex-dual function

$$\lambda(x,\epsilon) \equiv V_x(x,\epsilon) \quad G(\lambda(x,\epsilon)) \equiv x + \frac{\epsilon}{r^\epsilon}$$

- The dual HJB equation: for $0 < \lambda < \overline{\lambda}$,

$$rG(\lambda) = \frac{1}{2}\theta^2\lambda^2 G''(\lambda) + (\beta + \theta^2 - r)\lambda G'(\lambda) - \frac{1}{\gamma}\ln\lambda,$$

subject to

$$G(\overline{\lambda}) = \frac{\epsilon}{r^\epsilon}, \quad G'(\overline{\lambda}) = 0$$

Model 1: Solution

- Solution: for $0 < \lambda < \bar{\lambda}$,

$$G(\lambda) = -\frac{1}{\gamma r} \ln \lambda - \frac{\theta^2}{2\gamma r^2} \left(1 + \frac{2}{\theta^2}(\beta - r)\right) + B\lambda^{-\alpha^*},$$

where $-1 < \alpha^* < 0$ is the negative root of the following characteristic equation:

$$F(\alpha) \equiv -\frac{1}{2}\theta^2\alpha(\alpha - 1) + \alpha(\beta - r) + r = 0,$$

$$B = -\frac{\bar{\lambda}^{\alpha^*}}{\gamma r \alpha^*} > 0,$$

$$\bar{\lambda} = \exp\left\{ -\frac{\theta^2}{2r}\left(1 + \frac{2}{\theta^2}(\beta - r)\right) - \frac{1}{\alpha^*} - \gamma\epsilon \right\} > 0$$

Model 1: Optimal Strategies

- The optimal consumption and investment strategies

$$c = r\left[x + \frac{\epsilon}{r^{\epsilon}} + \frac{\theta^2}{2\gamma r^2}\left(1 + \frac{2}{\theta^2}(\beta - r)\right) - B\lambda^{-\alpha^*}\right],$$
$$\pi = \frac{\theta}{\gamma\sigma}\left(\frac{1}{r} + \alpha^* B\lambda^{-\alpha^*}\right)$$

- The optimal consumption is no longer affine in total wealth
- Levels of wealth affect stock investments

Model 2

- In the presence of a one-time Poisson shock, the labor income dynamics are: $\epsilon_0 = \epsilon > 0$,

$$d\epsilon_t = \mu^\epsilon \epsilon_{t-} dt - (1-k)\epsilon_{t-} dN_t,$$

  where $k \in [0, 1)$ is the income recovery parameter and N_t is the one-time Poisson shock with intensity $\delta > 0$

- The agent's income plummets from ϵ_{t-} to $k\epsilon_{t-}$ at the time when the disastrous Poisson shock occurs
- The positive income growth rate μ^ϵ

Model 2: Role of Insurance

- Without any consideration of a potential role of insurance in the income recovery in the aftermath of the income disaster, the agent's income would be completely wiped out reducing to nothing, i.e., $k = 0$
- Consider in a very reduced form the role of insurance for hedging the disastrous income shock
- With access to an insurance market to hedge against the income shock, the agent's income can be partly recovered at the rate of $0 < k < 1$, so that she receives $k\epsilon$ post disaster

Model 2: HJB Equation

- The HJB equation

$$\max_{(c,\pi)}\Big[-(\beta+\delta)V(x,\epsilon)+(rx-c+\epsilon)V_x(x,\epsilon)+\frac{1}{2}\pi^2\sigma^2V_{xx}(x,\epsilon)$$
$$+\pi\sigma\theta V_x(x,\epsilon)+\mu^{\epsilon}\epsilon V_{\epsilon}(x,\epsilon)-\frac{1}{\gamma}e^{-\gamma c}-\delta\frac{A}{\gamma r}e^{-\gamma r(x+k\epsilon/r^{\epsilon})}\Big]=0$$

- The post-disaster value function represented by the very last term on the right-hand side directly affects the pre-disaster value function, thus influencing optimal decisions pre disaster

Model 2: Convex-Duality Approach

- The dual HJB equation: for $0 < \lambda < \overline{\lambda}$,

$$rG(\lambda) = \frac{1}{2}\theta^2\lambda^2 G''(\lambda) + \Big\{\beta + \delta\Big(1 - \frac{A}{\lambda}e^{-\gamma r\left(G(\lambda) - \epsilon/r^\epsilon + k\epsilon/r^\epsilon\right)}\Big) + \theta^2 - r\Big\}\lambda G'(\lambda) - \frac{1}{\gamma}\ln\lambda,$$

subject to

$$G(\overline{\lambda}) = \frac{\epsilon}{r^\epsilon}, \quad G'(\overline{\lambda}) = 0$$

- The expected return compensation for the presence of the disastrous income shock and the disastrous income risk premium

$$\beta + \delta\Big(1 - \frac{A}{\lambda}e^{-\gamma r\left(G(\lambda) - \epsilon/r^\epsilon + k\epsilon/r^\epsilon\right)}\Big) + \theta^2 - r$$

Model 2: Optimal Strategies

- The optimal consumption and investment strategies

$$c = r\left[x + \frac{\epsilon}{r^{\epsilon}} + \frac{\theta^2}{2\gamma r^2}\left(1 + \frac{2}{\theta^2}(\beta + \delta - r)\right) - B\lambda^{-\alpha_\delta^*} + \mathrm{PS}\right],$$

$$\pi = \frac{\theta}{\gamma\sigma}\left(\frac{1}{r} + \alpha_\delta^* B\lambda^{-\alpha_\delta^*} + \alpha_\delta \mathrm{PS1} + \alpha_\delta^* \mathrm{PS2} - \mathrm{RD}\right),$$

where PS represents the precautionary savings driven by the disastrous income shock and it is given by

$$\mathrm{PS} = \mathrm{PS1} + \mathrm{PS2},$$

$$\mathrm{PS1} = \frac{2\delta(\alpha_\delta - 1)}{\theta^2(\alpha_\delta - \alpha_\delta^*)}\lambda^{-\alpha_\delta}\int_0^{\lambda}\mu^{\alpha_\delta - 2}\frac{A}{\gamma r}e^{-\gamma r\left(G(\mu) - \frac{\epsilon}{r^{\epsilon}} + \frac{k\epsilon}{r^{\epsilon}}\right)} > 0,$$

$$\mathrm{PS2} = \frac{2\delta(\alpha_\delta^* - 1)}{\theta^2(\alpha_\delta - \alpha_\delta^*)}\lambda^{-\alpha_\delta^*}\int_{\lambda}^{\overline{\lambda}}\mu^{\alpha_\delta^* - 2}\frac{A}{\gamma r}e^{-\gamma r\left(G(\mu) - \frac{\epsilon}{r^{\epsilon}} + \frac{k\epsilon}{r^{\epsilon}}\right)} < 0,$$

and RD represents the risk diversification demand driven by the disastrous income shock and it is given by

$$\mathrm{RD} = \frac{2\delta}{\theta^2\lambda}\frac{A}{\gamma r}e^{-\gamma r(x + k\epsilon/r^{\epsilon})} > 0$$

Model 3

- Thinking about large, negative income shocks as recurring events that repeat over time (e.g., the great depression, the 2008 global financial crisis, the recent COVID-19 pandemic), the income shocks are state dependent disasters that fluctuate in extreme events
- Consider a general Poisson jump process with state-dependent and stochastically time-varying disaster intensity δ_t (instead of constant intensity δ)
- The income dynamics ϵ_t are then evolved by: $\epsilon_0 = \epsilon > 0$,

$$d\epsilon_t = \mu^\epsilon \epsilon_{t-} dt - (1-k)\epsilon_{t-} dN_t^G,$$

where N_t^G is the Poisson jump process with state-dependent and time-varying intensity δ_t

Model 3 (Cont'd)

- State-dependent disastrous income shocks are modeled by a two-state Markov chain: the good state G and the bad state B
- For a small time period $(t, t+dt)$, the state switches from the good state G (B) to the bad state B (G) with probability $\phi^G dt$ ($\phi^B dt$) when the current state is G (B), and stays unchanged with the remaining probability $1-\phi^G dt$ ($1-\phi^B dt$)
- The intensity dynamics δ_t^i in the state i are: $\delta_0^i = \delta^i > 0$,

$$d\delta_t^i = -\delta^i \delta_t^i dt + b^i \delta_t^i dZ_t,$$

  where b^i is the volatility on the intensity growth rate and Z_t is a standard one-dimensional Brownian motion that is correlated with the market factor W_t considered in the stock price dynamics
- The negative intensity growth rate

Conclusion

- The low-probability, depression-like additional state in the agent's income caused by disastrous income risk significantly affects the agent's optimal choices
- Standard precautionary savings argument: consume less and save more
- The precautionary savings urn out to contribute to an increase in risky investments: the role of partial hedging against disastrous income risk by dynamically trading in the stock market
- The role of insurance for income recover post disaster allows the agent to consume more than with no access to insurance